347

347

OFFICE OF THE REGISTRAR OF BOOKS
15 MAY 1944
MADRAS

THE JOURNAL

OF THE

INDIAN MATHEMATICAL SOCIETY

NEW SERIES

2052

Volume VII **Nos. 1 & 2. March—June 1943**

CONTENTS

	Pages
V. Ganapathy Iyer: The Influence of Zeros on the Magnitude of Functions Regular in an Angle..	1—16
S. Minakshisundaram: On Expansion in Eigen Functions of Boundary Value Problems IV ..	17—24
K. Chandrasekharan: The Absolute Summability of Series of Eigen Functions ...	25—30
———: Bessel-Summability of the Product of Two Series ..	31—35
Ram Ballabh: Steady Uniplanar Superposable Fluid Motions ...	36—41
T. Venkata Rayudu: The Characters of the Classes $(n-k, k)$ of the Symmetric Group of Degree n.	42—45
B. R. Pasricha: Some Infinite Integrals involving Whittaker Functions ..	46—50
Hsien-Chung Wang: On the Projective Deformation of a Family of Elements of Contact ..	51—57
P. Kesava Menon: Identities in Multiplicative Functions ..	58—62

MADRAS:
PRINTED AT THE MADRAS LAW JOURNAL PRESS, MYLAPORE.

The Indian Mathematical Society

Founded in 1907

President: DR. F. W. LEVI (University of Calcutta).

Treasurer: L. N. SUBRAMANIAM, M.A. (Madras Christian College, Madras).

Secretaries: S. MAHADEVA IYER, M.A., L.T. (Presidency College, Madras).

M. R. SIDDIQUI (Osmania University, Hyderabad).

The Committee of Management includes the President, the Secretaries, the Treasurer and the following members:—

R. P. SHINTRE (Librarian), D. D. KOSAMBI, RAM BEHARI, A. NARASINGA RAO, A. N. SINGH, N. R. SEN, S. S. PILLAI and J. MACLEAN.

The Society publishes two quarterlies, the *Journal of the Indian Mathematical Society*, and the *Mathematics Student* which are published in March, June, September and December. The *Journal* is managed by the following Board of Editors:—

R. VAIDYANATHASWAMI (*Chairman*)

K. ANANDA RAO	C. RACINE
D. D. KOSAMBI	N. R. SEN
F. W. LEVI	M. R. SIDDIQUI
S. S. PILLAI	T. VIJAYARAGHAVAN

C. N. SRINIVASIENGAR (*Secretary*)

1. Papers intended for the *Journal* should be addressed to C. N. Srinivasiengar, D.SC., College of Engineering, Bangalore, and papers intended for the *Student* should be addressed to A. Narasinga Rao, M.A., D.SC., Annamalai University, Annamalai Nagar.

2. Authors of papers appearing in the *Journal* will receive 50 reprints free of charge. Extra reprints can be had at cost price, on intimation.

3. The Secretary should be addressed to for back numbers of both the Quarterlies, as well as by intending members and subscribers.

• Subscription for each of the Quarterlies is Rs. 6 per year. Single copy Rs. 1-8-0.

129992
-600 008.
5 MAY 1944
MADRAS

THE INFLUENCE OF ZEROS ON THE MAGNITUDE OF FUNCTIONS REGULAR IN AN ANGLE

BY

V. GANAPATHY IYER, *Annamalai University*

[Received 10 September 1942]

1. It is known* that the lattice-points in the complex plane

$$z = m + in, \quad m, n = 0, \pm 1, \pm 2, \ldots \tag{1}$$

have several interesting properties in relation to the values assumed by integral functions and functions regular in an angle at these points. The object of this paper is to show that the lattice-points (1) are only a special case of a large class of sequences which may be called pseudo-lattice points in relation to which also, integral functions and functions regular in an angle possess similar properties.

2. *Definitions and notation.* Let $z = re^{i\theta}$. We denote by $\Lambda(\alpha)$ the region

$$|\theta| \leqslant \alpha, \ (0 < \alpha \leqslant \pi)$$

where α is fixed in any particular discussion. Let $f(z)$ be a function regular in $\Lambda(\alpha)$ except perhaps at $z = 0$ where for simplicity, we suppose that $f(z)$ is continuous as $z \to 0$ from within $\Lambda(\alpha)$. We write

$$M\{r, f, \Lambda(\alpha)\} = \max |f(z)|$$

as z varies in the sector common to $\Lambda(\alpha)$ and the circle

* See M. L. Cartwright, *Proc. London Math Soc.* (2) 43, 26-32; A. Pfluger, *ibid* (2) 42, 305-15; J.M. Whittaker, *ibid.* (2) 37, 383-401, *Proc. Edinburgh Math. Soc.* 2 (1930), 111-28, *Interpolatory Function Theory* (Cambridge Math. Tracts, 33; 1935); V. Ganapathy Iyer, *Jour. Lond. Math. Soc.* 11 (1936), 247-9, 13 (1938), 91-4.

$|z|=r$. The order ρ, the type $k\{f, \Lambda(\alpha)\}$, and the Lindelof function $h(\theta, f)$ are defined by the relations

$$\rho = \overline{\lim_{r\to\infty}} \frac{\log \log M\{r, f, \Lambda(\alpha)\}}{\log r}$$

$$k\{f, \Lambda(\alpha)\} = \overline{\lim_{r\to\infty}} \frac{\log M\{r, f, \Lambda(\alpha)\}}{r^{\rho}}$$

$$h(\theta, f) = \overline{\lim_{r\to\infty}} \frac{\log |f(re^{i\theta})|}{r^{\rho}}, \quad |\theta| \leqslant \alpha$$

where ρ is finite. When $f(z)$ is an integral function we write $M(r, f) = M\{r, f, \Lambda(\pi)\}$ and we denote by $C(\rho, d)$ the class of all integral functions satisfying the condition

$$\overline{\lim_{r\to\infty}} \frac{\log M(r, f)}{r^{\rho}} < d.$$

2.1. Let $\{z_n\}$ be a sequence of points in the complex plane having infinity as its sole limit point.* We shall suppose that the points of the sequence are arranged according to non-decreasing moduli, the points with the same modulus being arranged according to their amplitude. We shall refer to the exponent of convergence† of the sequence simply as its order.

2.2. We now proceed to define the pseudo-lattice sequences. Let ρ and d be two *positive* numbers. Let $g(z)$ be an integral function. We say that $g(z)$ belongs to *the class* $M(\rho, d)$ if

$$\lim \frac{\log |g(z)|}{|z|^{\rho}} = d$$

as $|z| \to \infty$ outside a system of circles *the sum of whose radii is finite.* It is known that if $\{z_n\}$ denote the zeros of a function $g(z)$ of $M(\rho, d)$, then, [$n(r, \alpha, \beta)$ denoting the number of z_n in the sector $|z| = r$, $\alpha \leqslant \theta \leqslant \beta$],

* The points of $\{z_n\}$ need not all be distinct.

† The exponent of convergence of $\{z_n\}$ is the lower bound of numbers h such that $\Sigma |z_n|^{-h}$ converges.

$$\lim_{r\to\infty} \frac{n(r, \alpha, \beta)}{r^\rho} = \frac{d\rho}{2\pi}(\beta-\alpha)$$

so that the amplitudes of the zeros are everywhere dense in $0 \leqslant \theta \leqslant 2\pi$. We shall say that a sequence $\{z_n\}$ is *a pseudo-lattice sequence of order* ρ *and type* d if they are *the zeros* (*and the only zeros*) *of a function of* $M(\rho, d)$. We shall denote this by saying that *the sequence belongs to* $L(\rho, d)$.

2.3. We shall have to deal with a sub-class of $L(\rho,d)$. A function $g(z)$ is said *to belong to* $M_1(\rho, d)$ if (i) *it belongs to* $M(\rho,d)$ (ii) *its zeros* $\{z_n\}$ *are all simple and* (iii)

$$\lim_{n\to\infty} \frac{\log|g'(z_n)|}{|z_n|^\rho} = d. \tag{2}$$

We shall say that a sequence $\{z_n\}$ *belongs to* $L_1(\rho, d)$ if it constitutes *the zeros of a function of* $M_1(\rho, d)$. By definition the points of a sequence of $L_1(\rho, d)$ are all distinct.

2.4. If $g(z)$ belongs to $M(\rho, d)$, then

$$\lim \frac{\log|g(z)|}{|z|^\rho} = d \tag{3}$$

as $|z| \to \infty$ outside a system $C(g)$ of circles of total finite radius. The number of circles in $C(g)$ is infinite and their centres may be supposed to tend to infinity since otherwise (3) cannot hold for positive ρ and d. Now if C' be another system of circles with total finite radius, then (3) obviously holds outside $C'+C(g)$ so that the definition of $M(\rho, d)$ does not imply the uniqueness of the circles $C(g)$. But for a given function $g(z)$ of $M(\rho, d)$ we can determine a more or less unique system as follows. If we can suppress an infinity of circles of $C(g)$ and (3) still holds as $|z| \to \infty$ outside the remaining circles, then we suppress such circles. Thus we may suppose that the system of circles is the least possible in the sense that omission of an infinity of circles would render (3) untrue outside the remaining circles. Such a least system we call an *exceptional system of circles associated with* $g(z)$ *and denote it by* $E(g)$. The system of non-overlapping domains determined by $E(g)$

will be denoted by $D(g) = \{D_n(g)\}$. It may be noted that the diameter of $D_n(g)$ is bounded and tends to zero as $n \to \infty$. It is known that if $\{z_n\}$ belongs to $\overset{\circ}{L}_1(\rho, d)$, then the exceptional circles $E(g)$ associated with the corresponding function $g(z)$ of $M_1(\rho, d)$ can be taken to be the system of circles with centres at $z = z_n$ and radii $B|z_n|^{-h}$, where h is any fixed number greater than ρ and B is any positive constant*. In this case the exceptional circles will be *denoted by* $E(g, h)$, the corresponding non-overlapping domains by $D(g, h) = \{D_n(g, h)\}$ and we suppose that $B = 1$ without any loss of generality.

2.5 *Examples of pseudo lattice sequences.* By using the pseudo-periodic properties of the canonical product associated with lattice points (1) it can be easily shown† that (1) belongs to $L_1(2, \pi/2)$. The sequence

$$\{(2\rho d)^{-1/\rho} n^{2/\rho} e^{2\nu\pi i/n}\}, \quad \begin{pmatrix} n = 1, 2, 3, \ldots \\ \nu = 0, 1, \ldots, n-1 \end{pmatrix} \tag{4}$$

belongs to $L_1(\rho, d)$. The zeros of

$$\prod_{n=1}^{\infty} (1 - z^n/n^{\alpha n})^n \tag{5}$$

belong to $L(3/\alpha, \alpha/9)$. The zeros of

$$\prod_{n=1}^{\infty} \left\{1 - \frac{z^n}{n^{\alpha n}}\right\}\left\{1 - \frac{z^n}{(n+r_n)^{\alpha n}}\right\}, \tag{6}$$

where $r_n = \exp(-n^\lambda)$, $(\lambda > 0)$, belong to $L(2/\alpha, \alpha/2)$. The following theorem gives a general class of sequences belonging to $L_1(\rho, d)$.

THEOREM 1.‡ *Let $l(x)$ and $m(x)$ be two differentiable functions of x in $0 < x < \infty$, tending to zero as $x \to \infty$. Let*

$$xl'(x) \to 0, \; x \log x \, m'(x) \to 0$$

* See V. Ganapathy Iyer, Maximal Integral Functions, *Jour. Ind. Math. Soc.* (2) 4 (1940), 97-115.

† See J. M. Whittaker, *Proc. Edinburgh Math. Soc. loc. cit.*

‡ The proof that (4) belongs to L_1 (ρ, d) is found in *Trans. Amer. Math. Soc.* 42 (1937), 363-4. The proof for (5), (6) and Theorem 1 can be carried out on similar lines. See also V. Ganapathy Iyer, *Quart. Jour. Math.* (Oxford), 9 (1938), 206-15.

as $x \to \infty$. *Let* $\alpha, \eta, \lambda, \mu, L, M$ *be positive constants and let* $\{\lambda_n\}$, $(0 < \lambda_1 < \lambda_2 \ldots < \lambda_n \to \infty)$ *be a sequence defined by* $\lambda_n = Ln^\lambda\{1 + l(n)\}$. *Let* $\{\mu_n\}$, $(0 < \mu_1 \leqslant \mu_2 \ldots \leqslant \mu_n \to \infty)$ *be a sequence of integers defined by* $\mu_n = Mn^\mu\{1 + m(n)\}$.

Then the sequence

$$\{\eta\lambda_n^\alpha e^{2\nu\pi i/\mu_n}\}, \begin{pmatrix} n = 1, 2, 3, \ldots \\ \nu = 0, 1, 2, \ldots, \mu_n - 1 \end{pmatrix} \tag{7}$$

belongs to $L_1(\rho, d)$, *where*

$$\rho = \frac{\mu - 1}{\alpha\lambda}, \; d = \frac{M\alpha\lambda}{\eta^\rho(\mu+1)^2 L^{(\mu+1)/\lambda}}. \tag{8}$$

2.6. Given $\rho, d, L, M, \lambda, \mu$ the numbers α and η can be found from (8) so that (7) belongs to $L_1(\rho, d)$. We shall prove towards the end that if $\{z_n\}$ belongs to $L(\rho, d)$ then all sequences sufficiently near $\{z_n\}$ also belong to $L(\rho, d)$.

3. Before proceeding to prove the main theorems we shall state some lemmas. The first lemma collects together the main properties of $h(\theta, f)$.

Lemma 1.* *Let* $f(z)$ *be a function regular, of order* ρ *and finite type in* $\Lambda(\alpha)$. *Then we have*

(*a*) $h(\theta, f)$ *is a continuous function of* θ *in* $|\theta| \leqslant \alpha$;

(*b*) *given* ε *there is an* $r_0 = r_0(\varepsilon)$ *such that*

$$\log |f(re^{i\theta})| \leqslant \{h(\theta, f) + \varepsilon\} r^\rho$$

for $r \geqslant r_0$ *uniformly in* $|\theta| \leqslant \alpha$;

(*c*) *if* θ_1 *and* θ_2 *are such that* $|\theta_1| \leqslant \alpha, |\theta_2| \leqslant \alpha$, $0 < \theta_2 - \theta_1 < \pi/\rho$

* See E. C. Titchmarsh, *The Theory of Functions*, pp. 181-6.

If $h(\theta_0, f) = -\infty$ for some $\theta = \theta_0$ in $|\theta| \leqslant \alpha$, then $h(\theta, f) = -\infty$ for all $|\theta| < \alpha$. In this case continuity may fail at $\theta = \pm\alpha$, it being supposed that if $h(\theta, f) = -\infty$ for all $|\theta - \theta_0| \leqslant \delta$, then it is continuous at $\theta = \theta_0$.

and $$H_f(\theta, \theta_1, \theta_2) = A \cos \rho\theta + B \sin \rho\theta,$$

where A and B are determined from the equations

$$H_f(\theta_1, \theta_1, \theta_2) = h(\theta_1, f)\ ;\ H_f(\theta_2, \theta_1, \theta_2) = h(\theta_2, f)$$

then $$h(\theta, f) \leqslant H_f(\theta, \theta_1, \theta_2),\ \theta_1 \leqslant \theta \leqslant \theta_2;$$

(*d*) *if* $\alpha = \pi/2\rho$ *and* $h(\pi/2\rho, f) + h(-\pi/2\rho, f) < 0$ *then* $f(z) \equiv 0$.

3·1. By using the fact that the sum of the radii of circles of $E(g)$ is finite we easily deduce

LEMMA 2. *Let* $E(g)$ *be the exceptional circles of a function* $g(z)$ *of* $M(\rho, d)$. *Then given* $\lambda > 0$ *we can find* $r_0 = r_0(\lambda)$ *such that there is a circle* $|z| = r'$ *in the ring* $r - \lambda \leqslant |z| \leqslant r$ *which does not cut any circle of* $E(g)$, *for every* $r \geqslant r_0$.

3·2. Lemma 2, together with the fact that $M(r, g)$ is an increasing function of r leads to the following result in virtue of (3).

LEMMA 3. *A function of* $M(\rho, d)$ *is of order* ρ *and type* d.

4. Here we prove three theorems regarding functions regular in an angle having zeros at the points of a sequence of $L(\rho, d)$ lying in that angle. The significance of these theorems can be understood from the following considerations. The result in Lemma 1 (*c*) gives information regarding the order of magnitude of a function inside an angle, given the order of increase on the arms of the angle. It can be shown that this result is the best possible. But it is known that if there be a sufficient number of zeros inside the angle, the estimate obtained above can be considerably improved. So far, the known results* in this direction are confined to the case in which the zeros lie on the bisecting line of the angle or very near it. The case of Theorem 3 when the zeros lie at the lattice points (1) has also been discussed.† We

* See M. L. Cartwright, *Proc. Lond. Math. Soc.* (2) 32 (1930), 51-7.

† See A. Pfluger and J. M. Whittaker, *l.c.*

consider here the case in which the zeros lie at a sequence of $L(\rho, d)$. We consider separately three cases arising from the relation of the magnitude α of the angle $\Lambda(\alpha)$ to the order ρ of the function considered, viz. $\alpha < \pi/2\rho$, $\alpha = \pi/2\rho$, and $\alpha > \pi/2\rho$.

4.1. THEOREM 2. *Let $f(z)$ be a function, regular, of order ρ and finite type in $\Lambda(\alpha)$, $\alpha < \pi/2\rho$. Let $\{z_n\}$ be a sequence of $L(\rho, d)$. Let $f(z)$ have a zero of at least as high an order as* $g(z)$ at all those $z_n \subset \Lambda(\alpha)$. Then*

$$h(\theta, f) \leqslant H_f(\theta, -\alpha, \alpha) - d\left(\frac{\cos \rho\theta}{\cos \rho\alpha} - 1\right) \qquad (9)$$

for $|\theta| \leqslant \alpha$.

PROOF. We may suppose that $h(\theta, f)$ is not equal to $-\infty$ for any θ in $|\theta| \leqslant \alpha$, for, then $h(\theta, f) = -\infty$ for all θ in $|\theta| < \alpha$ and (9) is then trivial. We can also suppose without loss of generality that the lines $\theta = \pm\alpha$ cut only a finite number of circles of $E(g)$, since otherwise we can find an $\alpha' < \alpha$ and as near to it as we please such that the lines $\theta = \pm\alpha'$ have the desired property, and we can argue with the region $\Lambda(\alpha')$ and then let $\alpha' \to \alpha$ through a properly chosen sequence of values. All this is possible because $\alpha < \pi/2\rho$, $h(\theta, f)$ is continuous and Lemma 1 (c) can be applied to the whole angle $\Lambda(\alpha)$ as well as to a part of it. Now consider the function

$$F(z) = f(z)/g(z).$$

This is regular in $\Lambda(\alpha)$ by hypothesis. Outside the circles $E(g)$ we have

$$|F(z)| \leqslant \exp\{O(r^\rho)\}, \qquad (10)$$

in virtue of the hypothesis on $f(z)$ and the relation (3) for $g(z)$. Since the diameters of the corresponding $\{D_n(g)\}$ are bounded, it is easily seen that (10) holds throughout

* $g(z)$ is the function of $M(\rho, d)$ corresponding to $\{z_n\}$ in $L(\rho, d)$. This will be understood throughout this paper unless otherwise stated.

$\Lambda(\alpha)$ by the maximum modulus principle, if we note that all but a finite number of $D_n(g)$ lie entirely in $\Lambda(\alpha)$. So $F(z)$ is of order ρ and finite type in $\Lambda(\alpha)$. On $\theta = \pm\alpha$, we have

$$h(\alpha, F) = h(\alpha, f) - d;\ h(-\alpha, F) = h(-\alpha, f) - d,$$

because (3) holds and only a finite number of circles of $E(g)$ cut $\theta = \pm\alpha$. Applying Lemma 1 (*c*) to $F(z)$ we get

$$h(\theta, F) \leqslant H_F(\theta, -\alpha, \alpha).$$

But $$H_F(\theta, -\alpha, \alpha) = H_f(\theta, -\alpha, \alpha) - d\frac{\cos\rho\theta}{\cos\rho\alpha};$$

and $$h(\theta, f) \leqslant h(\theta, F) + d, \text{ by Lemma 3.}$$

Combining these we get (9).

4.2. For $\alpha = \pi/2\rho$ we can prove the following

THEOREM 3. *Let $f(z)$ be regular, of order ρ and finite type in $\Lambda(\pi/2\rho)$ and let*

(*a*) $h(\alpha, f) + h(-\alpha, f) < 2d$,

(*b*) $\{z_n\}$ *defined as in Theorem 2 satisfy the same condition in relation to $f(z)$ as in Theorem 2. Then $f(z) \equiv 0$.*

PROOF. We shall first prove the theorem on the assumption that the lines $\theta = \pm\pi/2\rho$ cut only a finite number of circles of $E(g)$. Arguing as in the proof of Theorem 2 we can prove that

$$F(z) = f(z)/g(z)$$

is regular, of order ρ and finite type in $\Lambda(\pi/2\rho)$. By (3),

$$h(\pi/2\rho, F) = h(\pi/2\rho, f) - d;\ h(-\pi/2\rho, F) = h(-\pi/2\rho, f) - d$$

so that

$$h(\pi/2\rho, F) + h(-\pi/2\rho, F) < 0.$$

So by Lemma 1 (*d*), we conclude that $F(z) \equiv 0$ and therefore $f(z) \equiv 0$. If, now, the hypotheses regarding the lines $\theta = \pm\pi/2\rho$ are not satisfied we proceed as follows. Since the sum of the radii of $E(g)$ is finite we can find a $z_0 > 0$ such that the lines through $z = z_0$ parallel to

$\theta = \pm\pi/2\rho$ have the desired property. W take $z = z_0$ as the new origin and denote by $Z = Re^{i\phi}$ the point $z = re^{i\theta}$ in the old system. Let

$$f_1(Z) = f(z),\ g_1(Z) = g(z),\ F_1(Z) = f_1(Z)/g_1(Z),\ |\phi| \leqslant \pi/2\rho.$$

Then $f_1(Z)$ satisfies* in $|\phi| \leqslant \pi/2\rho$ the conditions satisfied by $f(z)$ in $|\theta| \leqslant \pi/2\rho$, and $\phi = \pm\pi/2\rho$ have the property assumed in the first part of the proof. Hence $F_1(Z) \equiv 0$ and so $f(z) \equiv 0$. This proves Theorem 3.

4.2. We next consider the case $\alpha > \pi/2\rho$.

THEOREM 4. *Let $f(z)$ be a function regular in $\Lambda(\alpha)$, $\alpha > \pi/2\rho$. Let $\{z_n\}$ be a sequence of $L(\rho, d)$. Let the following conditions hold:—*

(a) The lines $\theta = \pm\alpha$ cut only a finite number of circles of $E(g)$,

$$(b)\ \overline{\lim_{r\to\infty}} \frac{\log M\{r, f, \Lambda(\alpha)\}}{r^\rho} < d,$$

(c) $h(\pm\alpha, f) < d$,

(d) for $z_n \subset \Lambda(\alpha), f(z_n) = 0$, the order of the zero being at least as high as that of $g(z)$ at $z = z_n$.

Then $f(z) \equiv 0$.

The condition (a) can be replaced by

(a′) $f(z)$ is of order ρ and finite type in $\Lambda(\alpha)$.

PROOF. Let

$$F(z) = f(z)/g(z).$$

By (a), (c), (d) and (3) we easily see that $F(z)$ is bounded on the lines $\theta = \pm\alpha$. By (b) we can find $\delta > 0$ and a sequence

$$0 < R_1 < R_2 \ldots < R_n \to \infty$$

such that on $|z| = R_n$

$$|f(z)| \leqslant \exp\{(d-\delta) R_n^\rho\}$$

* This follows from Lemma 1 (a) and (b) when $h(\theta, f)$ is finite in $|\theta| \leqslant \pi/2\rho$. Otherwise $h(\theta, f) = -\infty$ for $|\theta| \leqslant \pi/2\rho$ by the footnote to Lemma 1 and so $h(\pm\pi/2\rho, f_1) \leqslant h(\pm\pi/2\rho, f)$ in any case.

and by Lemma 2, we can find $\{R'_{.n}\}$ such that

$$R_n - 1 \leqslant R'_n \leqslant R_n$$

while none of the circles $|z| = R'_n$ cuts any circle of $E(g)$.

On $|z| = R'_n$ we have by (3), $[z \varepsilon \Lambda(\alpha)]$

$$|F(z)| \to 0, \; n \to \infty.$$

Hence by the maximum modulus principle, $F(z)$ is bounded in $\Lambda(\alpha)$. Taking $\sigma = \pi/2\alpha < \rho$ we see that

$$\overline{\lim_{r\to\infty}} \frac{\log M\{r, F, \Lambda(\alpha)\}}{r^\sigma} \leqslant 0$$

while*

$$\lim_{r\to\infty} \frac{\log |F(re^{\pm i\alpha})|}{r^\sigma} = -\infty$$

so that by Lemma 1 (d) with σ in place of ρ we conclude that $F(z) \equiv 0$ and therefore $f(z) \equiv 0$. If (a') holds then $h(\theta, f)$ is continuous in $\Lambda(\alpha)$ (or $h(\theta, f) = -\infty$ in $|\theta| < \alpha$) so that $\alpha' < \alpha$ can be found near enough to α so that $h(\pm\alpha', d < d$ while $\theta = \pm\alpha'$ satisfy the condition (a)† of this theorem and we can argue as before.

*We have

$$\frac{\log |F(re^{\pm i\alpha})|}{r^\sigma} = \frac{\log |F(re^{\pm i\alpha})|}{r^\rho} r^{\rho-\sigma},$$

and
$$\overline{\lim_{r\to\infty}} \frac{\log |F(re^{\pm i\alpha})|}{r^\rho} = h(\pm\alpha, f) - d < 0.$$

So
$$\lim \frac{\log |F(re^{\pm i\alpha})|}{r^\sigma} = -\infty.$$

† The assumption that the lines $\theta = \pm\alpha$ cut only a finite number of circles of $E(g)$ occurs occasionally in this paper. It can be shown that, if a $\{z_n\}$ of $L(\rho, d)$ $[L_1(\rho, d]$ is such that the order of those points of $\{z_n\}$ that lie in such domains of $\{D_n(g)\}$ $[D_n(g,h)]$ as are cut by a line $\theta = \theta_0$ is less than ρ for every $\theta = \theta_0$ in $0 \leqslant \theta \leqslant 2\pi$, then if all points of $\{z_n\}$ which lie in such domains as are cut by any finite number of lines $\theta = \theta_k$, $k = 1, 2, \ldots, p$ be omitted, the remaining points of $\{z_n\}$ still belong to $L(\rho, d)$ $[L_1(\rho, d)]$. When this is so, the assumption stated at the beginning can altogether be dispensed with. It is easily verified that this is true of the sequences (1), (4), (5), (6), and those found in Theorem 1.

4.3. Theorem 2 shows that the presence of zeros at the points of a sequence of $L(\rho, d)$ in $\Lambda(\alpha)$, $\alpha < \pi/2\rho$, considerably decreases the order of increase of a function of order ρ and finite type inside the angle. Theorems 3 and 4 show that if $\alpha \geqslant \pi/2\rho$ and $f(z)$ has zeros at the points of $L(\rho, d)$, the order of increase of $f(z)$ in $\Lambda(\alpha)$ must be at least as high as that of the corresponding $g(z)$ inside and on the arms of the angle unless $f(z) \equiv 0$. In the next theorem we suppose that $f(z)$ is small compared to $g(z)$ at the points of a sequence of $L_1(\rho, d)$ and arrive at a conclusion similar to that of Theorem 2.

4.4. THEOREM 5. *Let $f(z)$ be regular, of order ρ and finite type in $\Lambda(\alpha)$ where $\alpha < \pi/2\rho$. Let $\{z_n\}$ be a sequence of $L_1(\rho, d)$ and let*

(*a*) $h(\alpha, f) > d$ *and* $h(-\alpha, f) > d$,

(*b*) $\overline{\lim\limits_{n\to\infty}} \dfrac{\log |f(z_n)|}{|z_n|^\rho} < d$, $z_n \subset \Lambda(\alpha)$.

Then

$$h(\theta, f) \leqslant H_f(\theta, -\alpha, \alpha) - d\left(\frac{\cos \rho\theta}{\cos \rho\alpha} - 1\right)$$

for $|\theta| \leqslant \alpha$.

PROOF. The case when $h(\theta, f)$ is not finite is trivial as in Theorem 2. Supposing $h(\theta, f)$ to be finite, it is continuous in $|\theta| \leqslant \alpha$ and as in Theorem 2 we may suppose that the lines $\theta = \pm\alpha$ cut only a finite number of circles of $E(g, h)$. Let

$$F(z) = \frac{f(z)}{g(z)} - \sum_{z_n \subset \Lambda(\alpha)} \frac{f(z_n)}{g'(z_n)} \frac{1}{z - z_n}.$$

Remembering that the exceptional circles for a sequence of $L_1(\rho, d)$ can be chosen to be a system $E(g, h)$ with $h > \rho$*, we see that in virtue of (2)

$$\sum_{z_n \subset \Lambda(\alpha)} \frac{f(z_n)}{g'(z_n)} \frac{1}{z - z_n} = O(1) \qquad (11)$$

* See end of § 2.4.

as $|z| \to \infty$ on $\theta = \pm\alpha$. Also

$$\overline{\lim_{|z|\to\infty}} \frac{1}{|z|^\rho} \log \left| \frac{f(z)}{g(z)} \right| = h(\pm\alpha, f) - d, \tag{12}$$

on $\theta = \pm\alpha$. Therefore by (a), (11) and (12) give

$$h(\pm\alpha, F) \leqslant h(\pm\alpha, f) - d > 0.$$

We can now proceed as in Theorem 2 and conclude that $F(z)$ is of order ρ and finite type in $\Lambda(\alpha)$ so that by Lemma 1 (c) we get

$$h(\theta, F) \leqslant l(\theta) = H_f(\theta, -\alpha, \alpha) - d \frac{\cos \rho\theta}{\cos \rho\alpha}.$$

It is easily shown that (a) implies that

$$l(\theta) > 0. \tag{13}$$

Now

$$f(z) = g(z) \left\{ F(z) + \sum_{z_n \subset \Lambda(\alpha)} \cdot \frac{f(z_n)}{g'(z_n)} \frac{1}{z - z_n} \right\}$$

so that by (13) and the estimate for $F(z)$, we get for $r \geqslant r_0 = r_0(\varepsilon)$

$$\log |f(re^{i\theta})| \leqslant [l(\theta) + d + \varepsilon] r^\rho$$

outside the circles of $E(g, h)$ since

$$\sum_{z_n \subset \Lambda(\alpha)} \frac{f(z_n)}{g'(z_n)} \frac{1}{z - z_n} = o(1)$$

as $|z| \to \infty$ outside $E(g, h)$. Arguing as in Theorem 2 we see that the last inequality holds throughout $\Lambda(\alpha)$ if ε be replaced by 2ε for $r \geqslant r_0(2\varepsilon)$. This proves the theorem.

4·5. It must be noted that in Theorem 5, (12) and (13) can be derived only if (a) is true, since then alone will the order of the term $f(z)/g(z)$ predominate the rest of the terms in $F(z)$. It may also be noted that but for this restriction the result is the same as if we had assumed $f(z_n) = 0$ as in Theorem 2. But Theorem 5 holds only for

sequences of $L_1(\rho, d)$ while Theorem 2 is true of any sequence of $L(\rho, d)$.

5. We conclude with the proof of the following theorem which enables us to construct any number of sequences of $L(\rho, d)$ starting from a given sequence of $L(\rho, d)$, and which also shows that if $\{z_n\}$ belongs to $L(\rho, d)$ then all sufficiently near sequences also belong to $L(\rho, d)$.

THEOREM 6. *Let $\{u_n\}$ be a sequence of $L(\rho, d)$. Let $\{v_n\}$ be a sequence such that*

$$\sum_{n=1}^{\infty} |u_n - v_n| \text{ is convergent.} \tag{14}$$

*Then $\{v_n\}$ is a sequence of $L(\rho, d)$.**

5.1. We establish a preliminary lemma. We denote by $g_u(z)$ the function of $M(\rho, d)$ corresponding to $\{u_n\}$. Since the removal or the introduction of a finite number of points of a sequence of $L(\rho, d)$ still leaves it a sequence of $L(\rho, d)$, we may suppose that none of the points u_n and v_n is zero. For the same reason we may suppose in virtue of (14) for all $n \geqslant 1$,

$$\left.\begin{array}{l} U_n = |u_n| > \tfrac{1}{4}\,;\; V_n = |v_n| > \tfrac{1}{4} \\ r_n = |u_n - v_n| < \tfrac{1}{4}\,;\; V_n > U_n - \tfrac{1}{4} \end{array}\right\}. \tag{15}$$

We denote by $C(u,v)$ the system of circles with centres u_n and v_n and radii $\frac{1}{4} r_n$ so that the sum of their radii is finite. By (14) we see that ΣU_n^{-h} and ΣV_n^{-h} converge or diverge together so that $\{u_n\}$ and $\{v_n\}$ have the same order ρ. Let p be the least integer such that $\Sigma U_n^{-(p+1)}$ and $\Sigma V_n^{-(p+1)}$ converge. Then we see that the series

$$\lambda = \frac{1}{p} \sum_{n=1}^{\infty} \left(\frac{1}{u_n^p} - \frac{1}{v_n^p} \right)$$

* It may be noted that if $\{u_n\}$ belongs to $L_1(\rho, d)$ we cannot assert that $\{v_n\}$ belongs to $L_1(\rho, d)$ even if v_n's are distinct.

converges absolutely. Let $\sigma_u(z)$ and $\sigma_v(z)$ be the canonical products associated with $\{u_n\}$ and $\{v_n\}$ respectively so that

$$\sigma_u(z) = \prod_{n=1}^{\infty} \left(1-\frac{z}{u_n}\right) \exp\left\{\frac{z}{u_n}+\frac{z^2}{2u_n^2}+\ldots+\frac{z^p}{pu_n^p}\right\},$$

$$\sigma_v(z) = \prod_{n=1}^{\infty} \left(1-\frac{z}{v_n}\right) \exp\left\{\frac{z}{v_n}+\frac{z^2}{2v_n^2}+\ldots+\frac{z^p}{pv_n^p}\right\}.$$

We have

LEMMA 4. *Let $\{u_n\}$ and $\{v_n\}$ be as in Theorem 6. Let p, λ, $\sigma_u(z)$, $\sigma_v(z)$, $C(u, v)$ be defined as above. Then*

$$\log|\sigma_u(z)|-\log|\sigma_v(z)| = R(\lambda z^p)+o(|z|^\rho) \tag{16}$$

as $|z|\to\infty$ outside the circles $C(u, v)$, $R(\lambda z^p)$ meaning the real part of λz^p.

PROOF. We have*

$$p \leqslant \rho < p+1 \tag{17}$$

if we remember that $\Sigma U_n^{-\rho}$ diverges in virtue of the property of the zeros of functions of $M(\rho, d)$ mentioned in § 2.2. Let $|z| = r$ be large. Then by Lemma 2 and (14) we can find an η, $\frac{1}{2}<\eta<1$ such that the circles $|z| = r-\eta$ and $|z| = r+\eta$ have no point in common with $C(u, v)$. We have

$$\log|\sigma_u(z)|-\log|\sigma_v(z)|$$

$$= \sum_{n=1}^{\infty}\left\{\log|(1-z/u_n)/(1-z/v_n)|+R\left[\frac{z}{u_n}+\frac{z^2}{2u_n^2}+\ldots+\frac{z^p}{pu_n^p} -\frac{z}{v_n}-\frac{z^2}{2v_n^2}-\ldots-\frac{z^p}{pv_n^p}\right]\right\}. \tag{18}$$

We now divide the sum (18) into three parts S_1, S_2 and S_3 corresponding to the ranges

$$U_n,\ V_n < r-\eta;\ r-\eta < U_n,\ V_n < r+\eta;\ U_n,\ V_n > r+\eta$$

respectively, this division being possible because the circles $|z| = r-\eta$ and $|z| = r+\eta$ do not cut any circle of $C(u, v)$. By using the details explained above we have

*E. C. Titchmarsh, *l.c.* pp. 249-55.

$$|S_3| \leqslant \sum_{U_n, V_n > r+\eta} \left| -\sum_{m=p+1}^{\infty} \frac{z^m}{m}\left(\frac{1}{u_n^m} - \frac{1}{v_n^m}\right) \right|$$

$$\leqslant \sum_{U_n, V_n > r+\eta} \left\{ \frac{r^{p+1}}{(u_n - \frac{1}{4})^{p+1}} \cdot \frac{|u_n - v_n|}{\eta - \frac{1}{4}} \right\}^{*}$$

$$= O(1) = o(r^\rho). \qquad (19)$$

Next we have

$$S_1 = \sum_{U_n, V_n < r-\eta} \left\{ \log\left|\frac{u_n - z}{v_n - z}\right| + \log\left|\frac{v_n}{u_n}\right| + R\left(\frac{z}{u_n} + \dots + \frac{z^p}{p u_n^p}\right) - R\left(\frac{z}{v_n} + \dots + \frac{z^p}{p v_n^p}\right) \right\}.$$

Now

$$\sum_{U_n, V_n < r-\eta} \left| \log\left|\frac{u_n - z}{v_n - z}\right| \right| \leqslant \sum_{U_n, V_n < r-\eta} \log\left(1 + \frac{|u_n - v_n|}{\eta}\right) = O(1),$$

and

$$\sum_{U_n, V_n < r-\eta} \left| \log\left|\frac{v_n}{u_n}\right| \right| \leqslant O\left(\sum_1^{\infty} |u_n - v_n|\right) = O(1).$$

Also using (14), we get

$$\sum_{U_n, V_n < r-\eta} \left| z\left(\frac{1}{u_n} - \frac{1}{v_n}\right) + \dots + \frac{z^{p-1}}{p-1}\left(\frac{1}{u_n^{p-1}} - \frac{1}{v_n^{p-1}}\right) \right| = O(|z|^{p-1}) = o(|z|^\rho),$$

while

$$R \sum_{U_n, V_n < r-\eta} \frac{z^p}{p}\left(\frac{1}{u_n^p} - \frac{1}{v_n^p}\right) = R(\lambda z^p) + o(|z|^\rho), \ (p \leqslant \rho).$$

Combining these we get

$$S_1 = R(\lambda z^p) + o(|z|^\rho). \qquad (20)$$

Lastly, since the number of u_n and v_n in $r-\eta < |z| < r+\eta$ is $o(r^\rho)$ by the property of the zeros of a function of $M(\rho, d)$, we get

* We use the fact that $\frac{r}{U_n - \frac{1}{4}} < 1$ and $\Sigma |u_n - v_n|$ converges.

$$|S_2| \leqslant o(r^\rho). \max_{(r-\eta<U_n, V_n<r+\eta)} \left| z\left(\frac{1}{u_n}-\frac{1}{v_n}\right)+\ldots+\frac{z^p}{p}\left(\frac{1}{u_n^p}-\frac{1}{v_n^p}\right) \right|$$
$$+ \sum_{r-\eta<U_n, V_n<r+\eta} \left\{ \left| \log \left| \frac{u_n-z}{v_n-z} \right| \right| + \left| \log \left| \frac{v_n}{u_n} \right| \right| \right\}.$$

But in the range considered, $|\log|u_n/v_n|| = O(1)$ while, for z outside $C(u, v)$, $|\log|(u_n-z)/(v_n-z)|| \leqslant \log 5$ so that we get finally

$$S_2 = o(|z|^\rho). \tag{21}$$

From (19), (20) and (21) we get (16).

6.2. Proof of Theorem 6. *Case* (1). Let ρ be not an integer. Then $p < \rho$ and Lemma 4 gives

$$\log|\sigma_u(z)| - \log|\sigma_v(z)| = o(|z|^\rho) \tag{22}$$

outside the circles $C(u, v)$. Since $\{u_n\}$ are the zeros of $g_u(z)$, we see by a classical theorem* and Lemma 3, that

$$g_u(z) = e^{P(z)}\sigma_u(z),$$

where $P(z)$ is a polynomial of degree not exceeding ρ. Take

$$g_v(z) = e^{P(z)}\sigma_v(z).$$

Since

$$\lim \frac{\log|g_u(z)|}{|z|^\rho} = d$$

as $|z| \to \infty$ outside the circles $E(g_u)$, we get by (22)

$$\lim \frac{\log|g_v(z)|}{|z|^\rho} = d$$

as $|z| \to \infty$ outside the circles $E(g_u)$ and $C(u, v)$. Hence $\{v_n\}$ belongs to $L(\rho, d)$.

Case (2). If ρ is an integer, then $\rho = p$ since $\Sigma U_n^{-\rho}$ diverges as remarked above. Here we take

$$g_v(z) = \sigma_v(z) \exp\{P(z) + \lambda z^p\}$$

so that

$$\log|g_u(z)| - \log|g_v(z)|$$
$$= \log|\sigma_u(z)| - \log|\sigma_v(z)| - R(\lambda z^p) = o(|z|^\rho),$$

by Lemma 4 as $|z| \to \infty$ outside $E(g_u)$ and $C(u, v)$. This completes the proof.

* E. C. Titchmarsh, *l.c.* p. 250.

ON EXPANSION IN EIGENFUNCTIONS OF BOUNDARY VALUE PROBLEMS IV

BY

S. MINAKSHISUNDARAM

[Received 24 September, 1942]

1. In III* we studied the problem of summability of the Fourier series of an arbitrary function. This paper is devoted to answer the question: *Given the series* $\sum_1^\infty a_n \omega_n(x, y)$, *what is the necessary and sufficient condition that this series may be the Fourier series of a function integrable in D?*

1.1. We consider the following classes of functions defined in the closed domain $D+C$ which form Banach spaces with suitable norms

(i) L^q, $q \geqslant 1$: the space of measurable functions $f(x, y)$ the qth powers of whose moduli are integrable in D with

$$\|f\| = \left\{ \iint_D |f(x,y)|^q dx\, dy \right\}^{1/q}.$$

(ii) B: the space of functions $f(x, y)$ bounded and measurable in $D+C$ with $\|f\| = \max |f(x,y)|$.

(iii) C_0: the class of bounded functions $f(x, y)$ vanishing on C and continuous in D, with $\|f\|$ as in (ii).

(iv) C_2: the subspace of C_0 whose element $f(x,y)$ is differentiable twice with $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial^2 f}{\partial y^2} \varepsilon L^2$.

1.2. We also use the following notations as in III.

* *Jour. Ind. Math. Soc.* (2) 6 (1942), 153-67.

For $p > 0, k \geqslant 0$

$$f_p(r;\, x, y) = f_p(r) = \frac{p}{\pi r^2} \iint_{C_r} \left(1 - \frac{t^2}{r^2} \right)^{p-1} f(\xi, \eta)\, d\xi\, d\eta, \tag{1}$$

where $\quad (x-\xi)^2 + (y-\eta)^2 = t^2 \leqslant r^2$

$$\theta_p(r;\, x, y) = \theta_p(r) = \sum_{n=1}^{\infty} a_n \omega_n(x, y)\ \alpha_p(\lambda_n r) \tag{2}$$

and $$R_k(w;\, x, y) = R_k(w) = \sum_{\mu_n \leqslant w} \left(1 - \frac{\mu_n}{w} \right)^k a_n \omega_n. \tag{3}$$

2. Closure and completeness of the Eigenfunctions. The Fourier series of a function $g(x,y) \,\varepsilon\, C_2$ is absolutely and uniformly convergent in and on the boundary of D. So if

$$g(x,y) \sim \Sigma b_n \omega_n \tag{4}$$

and if δ is an arbitrary positive number, we can find an integer N so that for $n \geqslant N$

$$\left| g(x,y) - \sum_{\nu=1}^{n} b_\nu \omega_\nu \right| < \delta/2 \tag{5}$$

for all points (x,y) in $D + C$. Now if $f(x, y) \,\varepsilon\, C_0$ and is continuous in $D+C$, we can find a function $g(x,y) \,\varepsilon\, C_2$ so that

$$| f(x, y) - g(x, y) | < \delta/2. \tag{6}$$

From (5) and (6) it follows that

$$| f(x, y) - \sum^{n} b_\nu \omega_\nu | < \delta. \tag{7}$$

Therefore if we take a decreasing null sequence of positive numbers (δ_n), we can find a sequence of polynomials $g_n(x, y) = \sum_{\nu=1}^{N(n)} b^n_\nu \omega_\nu$, linear in ω_ν, so that

$$| g(x, y) - g_n(x, y) | < \delta_n \tag{8}$$

for all points (x, y) in $D + C$, that is

$$\lim_{n=\infty} \| f - g_n \| = 0. \tag{9}$$

(9) shows that, if C_0' denotes the subspace of C_0, whose elements $f(x, y)$ are continuous in the closed

600 008

domain $D+C$ vanishing on C, *the eigenfunctions* $\omega_n(x, y)$ *form a closed set in the space* C_0'.

2.1. Similarly if $f(x, y) \varepsilon L^q$, $1 \leqslant q < \infty$ we can find $g(x,y) \varepsilon C_2$ so that

$$\|f-g\| < \delta/2 \tag{10}$$

$$\|g-\sum^{N} b_\nu \omega_\nu\| < \delta/2. \tag{11}$$

From (10) and (11),

$$\|f-\sum^{N} b_\nu \omega_\nu\| < \delta. \tag{12}$$

Hence arguing as in (8) we can find polynomials $g_n(x, y)$ so that

$$\lim_{n=\infty} \|f-g_n\| = 0.$$

We have thus established that *the eigenfunctions* ω_n *form a closed set in the space* L^q, $q \geqslant 1$.

2.2. If $f(x, y) \varepsilon B$, and δ and η are two positive numbers we can find a function $g(x, y) \varepsilon C_2$ so that

$$|f-g| < \delta/2 \tag{13}$$

for all points (x, y) in D except for a set of points of measure $< \eta$. From (5) and (13) we see that

$$|f-\sum^{N} b_\nu \omega_\nu| < \delta$$

for all points (x, y) in D except for a set of points of measure $< \eta$. This shows that we can find a sequence of polynomials $g_n(x, y)$ converging *weakly* to $f(x, y)$, that is

$$\lim_{n=\infty} \iint_D (f-g_n)\, h\, dxdy = 0 \tag{14}$$

for every function $h \varepsilon L$ and $\|g_n\| = O(1)$. (15)

In other words $\omega_n(x, y)$ *form a weakly closed set in the space* B.

2.3. We shall show that $\omega_n(x, y)$ *form a complete set in the space* L. Let $h(x, y) \varepsilon L$ and let

$$\iint_D h\omega_n\, dxdy = 0 \tag{16}$$

for every $n = 1, 2, \ldots$ If $f(x, y) \varepsilon B$ we can find polynomials $g_n(x, y)$ converging weakly to $f(x, y)$, that is satisfying (14). But by (16), $\iint_D h\, g_n\, dx\, dy = 0$. Hence $\iint_D h\, f\, dxdy = 0$ for every $f \varepsilon B$, so that $h \equiv 0$. Thus (16) implies $h \equiv 0$ which means completeness of ω_n in L. Since the spaces L^q, $q \geqslant 1$, B and C_0 are subspaces of $L, (\omega_n)$ continue to be complete in these spaces as well. We have thus

THEOREM I. *The normal orthogonal eigenfunctions* $\omega_n(x, y)$ *form a closed and complete set in the space* L^q, $1 \leqslant q < \infty$ *and* C_0'. *They are complete but weakly closed in the space* B.

3. Classes of Functions and their Fourier Series.

Let D_r denote the domain contained in D such that the minimum distance between the boundaries of D and D_r is equal to $r > 0$. If A denotes one of the spaces L^q, $1 \leqslant q < \infty$; B and C_0, $\overline{A}$ denotes the subspace of A, which consists of functions defined in D_r for some $r > 0$ and vanishing outside D_r, the space $\overline{A}$ is clearly dense in A.

3.1. The Space C_0:

Let $f(x, y) \varepsilon C_0$. Then by Theorem A in III, $R_k(w\,; x, y)$ converges to $f(x, y)$ as $w \to \infty$ for $k > \frac{3}{2}$. In fact if $f(x, y) \varepsilon D_r$,

$$f_p(r\,; x, y) = \theta_p(r\,; x, y)$$

the series for $\theta_p(r)$ being absolutely convergent for all $r > 0$ if $p > 1$. We know that $f_p(r)$ converges to $f(x, y)$ uniformly in D_s, $s > 0$. Hence for $k > \frac{3}{2}$

$$R_k(w\,; x, y) \to f(x, y) \tag{17}$$

uniformly in D_s for every $s > 0$. In other words this convergence is uniform in D.

3.11* Also $R_k(w; x, y)$ is a linear transformation of f into itself so that

$$\| R_k(w; x, y) \| \leqslant M(w) \|f\|, \qquad (18)$$

where $M(w)$ is independent of f. We shall see that $M(w) = O(1)$ as $w \to \infty$. Since $M(w)$ is independent of f, it is enough if we consider a function $f \varepsilon \overline{C}_0$. If $p > 1$, $\theta_p(r)$ is also a linear transformation of f and

$$\| \theta_p(r) \| \leqslant T(r) \| f \| \qquad (19)$$

where $T(r)$ is bounded at ∞. As $r \to 0$, since $f \varepsilon \overline{C}_0$

$$\| \theta_p(r) \| = \| f_p(r) \| \leqslant \| f \|$$

so that there exists a constant t independent of r, so that

$$\| \theta_p(r) \| \leqslant t \| f \|, \qquad (20)$$

from which we deduce that

$$\| R_k(w) \| \leqslant m \| f \| \qquad (21)$$

where m is independent of w, for $k > \frac{3}{2}$.

3.12. It must be noted that if $f(x, y) \varepsilon L^q$, $q \geqslant 2$ or B, the above argument continues to hold good with $\|f\|$ denoting the norm in L^q or B as the case may be, and we can assert (21). But (17) ceases to be true.

3.13 Now let us suppose

$$| R_k(w; x, y) | = O(1) \qquad (22)$$

for all (x, y) in $D+C$ and

$$\lim_{w=\infty} R_k(w; x, y) \qquad (23)$$

exists uniformly in D. Then, if the limit function in (23) is $f(x, y)$, the conditions (22) and (23) imply that

* The argument in this section assumes some relations in III. That $T(r)$ is bounded at ∞, follows from III (31). The passage from (20) to (21) is effected as in III (32).

The symbol $\| \quad \|$ is used to denote the norm of the function in question in the space to which it belongs. Thus if f and g belong to two different spaces we still write $\| f \|$ and $\| g \|$ to denote the norms of the functions in the respective spaces. $R_k(w)$ is considered as belonging to the same space as f, from which it is derived.

$R_k(w; x, y)$ converges weakly to $f(x, y)$. Hence for every n

$$\lim_{w=\infty} \iint_D R_k(w)\, w_n(x, y)\, dx\, dy = \iint_D f\omega_n\, dx\, dy, \tag{24}$$

that is

$$a_n = \iint_D f\omega_n\, dx\, dy. \tag{25}$$

In other words $a_1, a_2, \ldots$ are the Fourier coefficients of $f(x, y)$ which by (23) is continuous in D.

3.14. Thus, from §§ 3.11 and 3.13, we have

THEOREM 2. *A necessary and sufficient condition that the series $\Sigma a_n \omega_n$ may be the Fourier series of a function $f(x, y)$ continuous in D is that (22) and (23) are fulfilled.*

3.15. We have also,

THEOREM 3. *A necessary and sufficient condition that $\Sigma a_n \omega_n$ may be the Fourier series of a function continuous in the closed domain $D+C$ and vanishing on C is that*

$$\lim_{w=\infty} R_k(w; x, y)$$

exists uniformly in the closed domain $D+C$.

The proof of this theorem is exactly the same as that of Theorem 6 in § 3.31 when $\|f\|$ is used to denote the norm in C'_0.

3.2. THE SPACE B. From the remark in § 3.12 it is clear that if $f(x, y) \varepsilon B$,

$$\| R_k(w; x, y) \| = O(1) \text{ as } w \to \infty. \tag{26}$$

Conversely if $\Sigma a_n \omega_n$ is such that (26) is fulfilled, there exists a function $f(x, y) \varepsilon B$, to which $R_k(w)$ converges weakly. Therefore (24) and (25) hold good, which proves that $\Sigma a_n \omega_n$ is the Fourier series of $f(x, y)$. Hence

THEOREM 4. *In order that $\Sigma a_n \omega_n$ may be the Fourier series of a function $f(x, y) \varepsilon B$, it is necessary and sufficient that (26) holds good.*

3·3 THE SPACE L^q, $1 \leqslant q < \infty$. Let q' be such that $1/q + 1/q' = 1$ so that if $q = \infty$ or 1, $q' = 1$ or ∞. Now, if $2 \leqslant q \leqslant \infty$, by the remark in § 3.12

$$\| R_k(w;\, x, y) \| = O(1) \tag{26}$$

and if $1 \leqslant q \leqslant 2$ we take a function $f(x, y) \,\varepsilon\, L^q$ and an arbitrary function $g(x, y) \,\varepsilon\, L^{q'}$, with $g(x, y) \sim \Sigma b_n \omega_n$.

Since by (21)

$$\left\| \sum_{\mu_n \leqslant w} \left(1 - \frac{\mu_n}{w}\right)^k b_n \omega_n \right\| \leqslant m \, \| g \|, \; k > \tfrac{3}{2},$$

$$\left| \sum_{\mu_n \leqslant w} \left(1 - \frac{\mu_n}{w}\right)^k a_n b_n \right| \leqslant m \, \| f \| \cdot \| g \|, \tag{27}$$

that is

$$\left| \iint_D R_k(w;\, x, y) \; g(x, y) \; dx \; dy \right| \leqslant m \, \| f \| \, . \, \| g \|. \tag{28}$$

Since $g(x, y)$ is arbitrary it follows that

$$\| R_k(w;\, x, y) \| \leqslant m \, \| f \|.$$

Thus (26) is fulfilled if $f(x, y) \,\varepsilon\, L^q$, $1 \leqslant q \leqslant 2$. Conversely if (26) is fulfilled, then provided $q > 1$, there exists a function $f(x, y) \,\varepsilon\, L^q$ to which $R_k(w)$ converges weakly, in the space L^q. Hence (24) and (25) follow. Thus we have

THEOREM 5. *In order that* $\Sigma a_n \omega_n$ *may be the Fourier series of a function* $f(x, y) \,\varepsilon\, L^q$, $1 < q \leqslant \infty$, *it is necessary and sufficient that* (26) *is fulfilled.*

3·31. Theorem 5 is not necessarily true if $q = 1$. While (26) is necessary, it is not sufficient. We have, however, the following theorem true for $1 \leqslant q < \infty$.

THEOREM 6. *A necessary and sufficient condition that* $\Sigma a_n \omega_n$ *may be the Fourier series of a function* $f(x, y) \,\varepsilon\, L^q$, $1 \leqslant q < \infty$, *is that*

$$\lim_{w = \infty} \| R_k(w) - f \| = 0, \; k > \tfrac{3}{2}. \tag{29}$$

PROOF. Let us write

$$R_k(w; f) = \sum_{\mu_n \leqslant w} (1-\mu_n/w)^k a_n \omega_n$$

so as to point out that the right side is derived from the function $f(x, y)$. Since the normal orthogonal eigen-functions form a closed set in L^q, we can determine a sequence of polynomials linear in ω_ν, namely $g_n(x, y)$, so that $\lim_{n=\infty} \| f-g_n \| = 0$. Now

$$\begin{aligned} R_k(w; f)-f &= R_k(w; f)-R_k(w; g_n) + R_k(w; g_n)-g_n+g_n-f \\ &= R_k(w; f-g_n)+R_k(w; g_n)-g_n+g_n-f, \end{aligned}$$

so that

$$\begin{aligned} \| R_k(w; f)-f \| &\leqslant \| R_k(w; f-g_n) \| + \| R_k(w; g_n)-g_n \| + \| f-g_n \| \\ &\leqslant m \| f-g_n \| + \| R_k(w; g_n)-g_n \| + \| f-g_n \| \\ &= (m+1) \| f-g_n \| + \| R_k(w; g_n)-g_n \|. \end{aligned}$$

Letting $w \rightarrow \infty$ first and then $n \rightarrow \infty$, we verify (29). Thus the necessity of the condition (29) is proved. Sufficiency follows from the fact that (29) implies (24) and therefore (25).

3.32. It is obvious that the following is just a re-statement of Theorem 6.

THEOREM 7. *In order that* $\Sigma a_n \omega_n$ *may be the Fourier series of a function* $f(x, y) \varepsilon L^q$, $1 \leqslant q < \infty$, *it is necessary and sufficient that*

$$\lim_{w, w'=\infty} \| R_k(w; x, y) - R_k(w'; x, y) \| = 0. \qquad (30)$$

THE ABSOLUTE SUMMABILITY OF SERIES OF EIGENFUNCTIONS

BY

K. CHANDRASEKHARAN, *Madras University*

[Received 24th July, 1942]

The problem of summability of series of eigenfunctions of boundary-value problems has been solved by Dr. S. Minakshisundaram*, by making use of the method of Bessel-summability developed by me in a paper† which is not yet published. The object of this note is to answer the corresponding question of absolute summability.

1. Let D be a bounded open domain, simply or multiply connected, whose boundary C is composed of a finite number of regular curves in the x, y plane, and let $\omega_n(x, y)$ be the eigenfunctions (normal and orthogonal) with the corresponding eigenvalues $\mu_n > 0, n = 1, 2, \ldots$ of the boundary-value problem

$$\triangle\omega + \mu\omega = \frac{\partial^2\omega}{\partial x^2} + \frac{\partial^2\omega}{\partial y^2} + \mu\omega = 0, \ \omega(x, y) = 0 \text{ on } C.$$

We assume that the eigenvalues μ_n have been arranged in increasing order of magnitude.

Let $f(x, y)$ be a function defined in D and integrable L^2, and let us write

$$f(x, y) \sim \Sigma a_n\omega_n(x, y), \ a_n = \iint_D f\omega_n \, dx \, dy.$$

The series $\Sigma a_n\omega_n(x, y)$ is called the Fourier series of $f(x, y)$. We assume $f(x, y)$ to be such that, if C_r is a circle

* S. Minakshisundaram : *Jour. Ind. Math. Soc.* (2) 6 (1942), 153-67.

† K. Chandrasekharan : *The Bessel Summation of Series*, to appear in *Proc. Ind. Acad. Sc.* 17 (1943). This and other papers quoted here form part of a thesis submitted for the M. Sc. degree of the University of Madras.

of radius $r > 0$, and centre (x, y) in D, so that C_r lies wholly in $D, f(x, y)$ is integrable on the circumference of C_r.

Therefore, we may write

$$\phi(r; x, y) = \phi(r) = \frac{1}{2\pi}\int_0^{2\pi} f(x+r\cos\theta, y+r\sin\theta)\, d\theta.$$

Let

$$\psi_p(t) = \frac{2\Gamma(p+1)}{\Gamma(p)}\int_0^t (t^2-u^2)^{p-1} u\phi(u)\, du,$$

$$a_{n\omega_n} = b_n,\ \sqrt{\mu_n} = \lambda_n,\ s(x) = \sum_{\nu=0}^{n} b_\nu, \text{ for } n \leqslant x < n+1, \text{ and}$$

$$s^{(r)}(x) = \frac{1}{\Gamma(r)}\int_0^x (x-t)^{r-1} s(t)\,dt,\ s^{(0)}(x) = s(x)\,;\ s_m = \sum^{m} b_r = s(m).$$

The series Σb_n is absolutely summable (λ, r), if

$$\int_0^\infty \left| \frac{d}{du}\frac{s^r(u)}{u^r} \right| du < \infty.$$

We establish the following results.

THEOREM A. *If the series* $\sum_{n=0}^{\infty} a_{n\omega_n}$ *is absolutely summable* (λ, r), *then* $\psi_\mu(t)/t^{2\mu}$ *will be of bounded variation in some finite interval enclosing the origin, provided* $\mu > \max(1, r+\frac{1}{2})$.

THEOREM B. *If* $\psi_\mu(t)/t^{2\mu}$ *is of bounded variation in some finite interval enclosing the origin, then the series* $\Sigma a_{n\omega_n}$ *is absolutely summable* (λ, r), *for* $r > \max(\frac{5}{2}, \mu+\frac{1}{2})$.

2. Preliminary Lemmas.

LEMMA 1. $\omega(x, y) = O(\lambda_n^{\frac{1}{2}}) = O(n^{\frac{1}{4}})$.*

LEMMA 2. *If* (x,y) *is an interior point of* D, $s(\omega) = O(\omega^{3/2})$.

LEMMA 3. *If* $s(\omega) = O(\omega^{3/2})$, *and* $s^r(\omega) = s\omega^r + o(\omega^r)$, *then* $s^k(\omega) = s\omega^k + O(\omega^{\frac{3}{2}(1-k/r)+k})$, *for* $0 < k \leqslant r$, *where* $s =$ *sum of the series in question.*

LEMMA 4. $D_x^r[\alpha_\mu(x)] = O(x^{-\mu-\frac{1}{2}})$, *for any integer* $r > 0$, *as* $x \to \infty$.

LEMMA 5. $\psi_\mu(t)/t^{2\mu} = \Sigma a_{n\omega_n}\alpha_\mu(\lambda_n t)$, *if* $\mu > \frac{1}{2}$, *the series being absolutely convergent for* $t > 0$ *such that* C_t *is in* D.

* See S. Minakshisundaram, *loc. cit.*

* LEMMA 6. *If* Σa_n *is summable* $|\lambda^2, \alpha|$, *then it is also summable* $|\lambda, \alpha|$.

3. PROOF OF THEOREM A.

$$\sum_0^m b_n \alpha_\mu(\lambda_n t) = \sum_0^{m-1} s_\nu[\alpha_\mu(\lambda_\nu t) - \alpha_\mu(\lambda_{\nu+1} t)] + s_m \alpha_\mu(\lambda_m t),$$

by Abel's method of partial summation. Now,

$$\alpha_\mu(\lambda_\nu t) - \alpha_\mu(\lambda_{\nu+1} t) = -\int_{\lambda_\nu}^{\lambda_{\nu+1}} \frac{d}{dx} \alpha_\mu(xt)\, dx,$$

so that

$$s_\nu[\alpha_\mu(\lambda_\nu t) - \alpha_\mu(\lambda_{\nu+1} t)] = -\int_{\lambda_\nu}^{\lambda_{\nu+1}} s(x) \frac{d}{dx} \alpha_\mu(xt) dx.$$

Therefore,

$$\sum_0^m b_n \alpha_\mu(\lambda_n t) = s_m \alpha_\mu(\lambda_m t) - \int_{\lambda_0}^{\lambda_m} s(x) \frac{d}{dx} \alpha_\mu(xt) dx.$$

Integration by parts, h times, where $h = [r]$, gives

$$\sum_0^m b_n \alpha_\mu(\lambda_n t) = s_m \alpha_\mu(\lambda_m t) - s^{(1)} \frac{d}{d\lambda_m} \alpha_\mu(\lambda_m t) + s^{(2)} \frac{d^2}{d\lambda_m^2} \alpha_\mu(\lambda_m t) - \dots$$
$$+ (-1)^h s^{(h)}(\lambda_m) \frac{d^h}{d\lambda_m^h} \alpha_\mu(\lambda_m t) + (-1)^{h+1} \int_0^{\lambda_m} s^h(x) \,.\, D_x^{h+1} \alpha_\mu(xt) dx. \tag{3.1}$$

From Lemmas 1 to 4, we have,

(i) If $k > 3/2$,

(*a*) $s^k(x) = o(x^k)$, if $k \geqslant r$.

(*b*) $s^k(x) = o(x^r)$, if $r - 3/2 \leqslant k < r$.

(*c*) $s^k(x) = O(x^{k+3/2})$, if $0 \leqslant k < r - 3/2$.

(ii) If $k \leqslant 3/2$,

(*a*) $s^k(x) = o(x^k)$, if $k \geqslant r$.

(*b*) $s^k(x) = O(x^{3/2})$, if $k < r$.

(iii) $D_x^k \alpha_\mu(xt) = O(x^{-(\mu+\frac{1}{2})})$, for $t > 0$, as $x \to \infty$.

Hence, if $k \geqslant 3/2$,

* K. Chandrasekharan : *Jour. Ind. Math. Soc.* 6 (1942), 168-80.

$$s^k(x)\, D_x^k \alpha_\mu(xt) = \begin{cases} O\left(\frac{1}{x^{\mu-1-k}}\right) & \text{if } 0 \leqslant k < r - \frac{3}{2} \\ O\left(\frac{1}{x^{\mu+\frac{1}{2}-r}}\right) & \text{if } r - \frac{3}{2} \leqslant k < r \\ O\left(\frac{1}{x^{\mu+\frac{1}{2}-k}}\right) & \text{if } k \geqslant r. \end{cases} \tag{3.2}$$

Hence,

$$s^k(x)\,.\, D_x^k\, \alpha_\mu(xt) = o(1), \text{ if } \mu > r - \tfrac{1}{2},\ k \leqslant h. \tag{3.3}$$

Similarly if $k < 3/2$,

$$s^k(x)\ D_x^k \alpha_\mu(xt) = o(1), \text{ if } \mu > \max\ (1,\ r - \tfrac{1}{2}).$$

Hence, if $\mu > \max\ (1, r - \frac{1}{2})$,

$$\sum_0^m b_n \alpha_\mu(\lambda_n t) = o(1) + (-1)^{h+1} \int_0^{\lambda_m} s^h(x)\,.\, D_x^{h+1} \alpha_\mu(xt)\ dx$$

$$= o(1) + \frac{(-1)^{h+1}}{\Gamma(1+h-r)} \int_0^{\lambda_m} D_x^{h+1}\ \alpha_\mu(xt)\ dx \int_0^x (x-u)^{h-r} s^{(r-1)}(u)\ du$$

$$= o(1) + \frac{(-1)^{h+1}}{\Gamma(1+h-r)} \int_0^{\lambda_m} s^{r-1}(u)\ du \int_u^{\lambda_m} (x-u)^{h-r}\,.\, D_x^{h+1} \alpha_\mu(xt)\ dx.$$

From (3.1), we obtain,

$$\sum_0^m b_n\ \alpha_\mu(\lambda_n\ t) = o(1) + \frac{(-1)^{h+1}}{\Gamma(1+h-r)} \int_0^{\lambda_m} s^{r-1}(u)\ du$$

$$\times \int_u^\infty (x-u)^{h-r}\,.\, D_x^{h+1}\ \alpha_\mu(xt)\ dx \tag{3.4}$$

for $\mu > \max\ (1,\ r + \frac{1}{2})$, since it can be seen that

$$\int_0^{\lambda_m} s^{(r-1)}(u)\ du \int_{\lambda_m}^\infty (x-u)^{h-r}\,.\, D_x^{h+1}\ \alpha_\mu(xt)\ dx \to 0 \text{ as } \lambda_m \to \infty,$$

provided $\mu > h + \frac{1}{2}$.

Letting $m \to \infty$ in (3.4), we obtain for $\mu > \max\ (1,\ r + \frac{1}{2})$

$$\sum_0^\infty b_n \alpha_\mu(\lambda_n t) = \frac{(-1)^{h+1}}{\Gamma(1+h-r)} \int_0^\infty s^{(r-1)}(u)\, du \int_u^\infty (x-u)^{h-r}. D_x^{h+1} \alpha_\mu(xt)\, dx. \tag{3.5}$$

Now, let

$$\chi_\mu(t) = \int_0^\infty s^{(r-1)}(u)\, du \int_u^\infty (x-u)^{h-r}. D_x^{h+1} \alpha_\mu(xt)\, dx. \tag{3.6}$$

Set

$$\mathcal{J}_z(t) = \int_z^\infty (x-z)^{h-r} D_x^{h+1} \alpha_\mu(xt)\, dx, \quad V_u(t) = \int_u^\infty z^r \frac{d}{dz} \mathcal{J}_z(t)\, dz.$$

Then, it can be proved that

$$\mathcal{J}_z(t) = O(z^{-\mu-\frac{1}{2}}), \text{ as } z \to \infty \tag{3.7}$$

$$V_u(t) = O(u^{r-\mu-\frac{1}{2}}), \text{ as } u \to \infty \tag{3.8}$$

$$\begin{aligned} |V_u'(t)| &= O(u^{r-\mu+\frac{1}{2}}\, t^{r-\mu-\frac{1}{2}}), \text{ if } |ut| \geqslant 1 \\ &= O(u^r t^{r-1}), \text{ if } |ut| < 1. \end{aligned} \tag{3.9}$$

Hence by integrating (3.6) by parts, we get

$$\chi_\mu(t) = \int_0^\infty \frac{d}{du}\frac{s^r(u)}{u^r} \,.\, V_u(t)\, du.$$

From (3.9), we observe that $V_u'(t) = O(1)$, as $u \to \infty$, if $t \geqslant \varepsilon$, where ε is an arbitrary positive number; and $\left|\frac{d}{du}\frac{s^r(u)}{u^r}\right|$ is integrable by hypothesis, so that we have,

$$\chi_\mu'(t) = \int_0^\infty \frac{d}{du}\frac{s^r(u)}{u^r} \cdot \frac{d}{dt} V_u(t)\, du. \tag{3.10}$$

Therefore,

$$\int_0^1 |\chi_\mu'(t)|\, dt \leqq \int_0^1 dt \int_0^\infty \left|\frac{d}{du}\frac{s^r(u)}{u^r}\right| \left|\frac{d}{dt} V_u(t)\right| du. \tag{3.11}$$

From (3.9), $\int_0^1 |V_u'(t)|\, dt = \int_0^{1/u} + \int_{1/u}^1 = O(1)$, uniformly in u, for $\mu > r + \frac{1}{2}$. Also

$$\int_0^\infty \left|\frac{d}{du}\frac{s^r(u)}{u^r}\right| du < \infty.$$

Hence, $\chi_\mu(t)$ is of bounded variation in (0, 1); thus from (3.5) and Lemma (5) we see $\psi_\mu(t)/t^{2\mu}$ is of bounded variation in (0, 1).

4. Proof of Theorem B.

$\phi_\mu(t) = \Sigma a_n \omega_n \alpha_\mu(\lambda_n t)$ is of bounded variation in (0, δ), say.

$$|\phi_\mu'(t)| = \left|\sum a_n \omega_n \frac{\mathcal{J}_{\mu+1}(\lambda_n t)}{(\lambda_n t)^\mu} \cdot \lambda_n \right| \leqq \sum \left| a_n \omega_n \frac{\mathcal{J}_{\mu+1}(\lambda_n t)}{(\lambda_n t)^\mu} \cdot \lambda_n \right|$$

$$\leqslant \sum | a_n \omega_n \alpha_{\mu+1}(\lambda_n t)\, \lambda_n^2 t | \leqslant \left\{ \sum a_n^2 \sum \omega_n^2 \alpha_{\mu+1}^2(\lambda_n t) \lambda_n^4 t^2 \right\}^{1/2}$$
$$\leqslant \left\{ \sum O(n^{\frac{1}{2}}) O\left(\frac{1}{\lambda_n^{2\mu-3}}\right) \right\}^{1/2} = \left\{ \sum O\left(\frac{1}{\lambda_n^{2\mu-2}}\right) \right\}^{1/2}.$$

Therefore, the series for $\phi_\mu'(t)$ will be absolutely convergent for every $t > 0$, provided $\mu > 2$, in which case it represents a function continuous at ∞.

By hypothesis, $\phi_\mu'(t)$ is of bounded variation in $(0, \delta)$, and if $\mu > 2$, it is also of bounded variation in (δ, ∞).

Let
$$F(\omega) = \omega^{2\mu+2} \int_0^\infty \alpha_\nu(\omega t)\, \phi_\mu(t)\, t^{2\mu+1} dt. \tag{4.1}$$

Then, it is easily proved that*
$$\lim_{\omega\to\infty} F(\omega) = \lim_{t\to 0} \phi_\mu(t) = 0,$$
and from the Theory of Bessel functions, we have,
$$F(\omega) = \sum_{\lambda_n < \omega} b_n \left(1 - \frac{n^2}{\omega^2}\right)^{\nu-\mu-1}, \quad \nu > 2\mu + \tfrac{3}{2}.$$

From (4.1), $F'(\omega) = -\omega^{2\mu+1} \int_0^\infty t^{2\mu+1} \phi_\mu'(t)\, \alpha_\nu(\omega t)\, dt.$

Hence, if $\nu > 2\mu + 1\frac{1}{2}$
$$\int_0^\infty | F'(\omega) |\, d\omega \leqslant \int_0^\infty t^{2\mu+2} | \phi_\mu'(t) |\, dt \int_0^\infty \omega^{2\mu+1} | \alpha_\nu(\omega t) |\, d\omega. \tag{4.2}$$

Now, by dividing the interval $(0, \infty)$ into $(0, 1/t)$ and $(1/t, \infty)$, we can prove that
$$\int_0^\infty \omega^{2\mu+1} | \alpha_\nu(\omega t) |\, d\omega = O\left(\frac{1}{t^{2\mu+2}}\right).$$

Hence from (4.2),
$$\int_0^\infty | F'(\omega) |\, d\omega < \infty,$$
i.e. Σb_n is summable $| \lambda_n^2, \nu - \mu - 1 |$ for $\nu > 2\mu + 1\frac{1}{2}$, $\mu > 2$, or summable $| \lambda_n^2, r |$ for $r > \max(\mu + \frac{1}{2}, 2\frac{1}{2})$. By Lemma 6 it follows that Σb_n is summable $| \lambda_n, r |$ for $r > \max(\mu + \frac{1}{2}, 2\frac{1}{2})$.

Remark. *If $f(x, y)$ is a continuous function of bounded variation, then the series will be absolutely summable (λ, r) at every interior point.*

* K. Chandrasekharan. *Bull. Calcutta Math. Soc.* 34 (1942), 187-96.

BESSEL-SUMMABILITY OF THE PRODUCT OF TWO SERIES

BY

K. CHANDRASEKHARAN, *University of Madras*

[Received 11 March, 1943]

1. The method of Bessel summation of series which was developed by me in a previous paper* is applied here to study the summability of the product of two given series. This product, however, is not the well-known Cauchy-product; I call it the Bessel-product. It is defined thus:

The Bessel-product of two series Σa_μ *and* Σb_ν *is equal to* Σc_n, *where* $c_n = \sum_{n=\sqrt{\mu^2+\nu^2}} a_\mu b_\nu$.

The exact connection between the summability of the Bessel-product and that of the Cauchy-product remains an open question.

2. Let $\mathcal{J}_\mu(t)$ be the Bessel function of order μ, viz.

$$\mathcal{J}_\mu(t) = \frac{t^\mu}{2^\mu\Gamma(\mu+1)}\left\{1 - \frac{t^2}{2(2\mu+2)} + \frac{t^4}{2.4(2\mu+2)\ (2\mu+4)} - \dots\right\}.$$

Let
$$\alpha_\mu(t) = 2^\mu\Gamma(\mu+1)\ \mathcal{J}_\mu(t)/t^\mu.$$

DEFINITION. *A series* Σa_n *is said to be summable* $\mathcal{J}_\mu$ *to the sum* s, *if* $\sum_0^\infty a_n\alpha_\mu(nt)$ *converges for real* $t > 0$, *and if* $\lim_{t\to 0} \phi_\mu(t) = s$, *where* $\phi_\mu(t) = \sum_0^\infty a_n\alpha_\mu(nt)$.

In what follows, we assume that $\Sigma a_n\alpha_\mu(nt)$ converges in such a manner as to permit integration of the series term by term, if we are given that Σa_n is summable $\mathcal{J}_\mu$.

* To appear shortly in the *Proc. Ind. Acad. Sci.*

If the a_n's are Fourier coefficients, this condition will always be realized.

3. THEOREM 1. *If Σa_n is summable J_μ, and Σb_n is summable J_ν, then the Bessel-product Σc_r is summable $J_{\mu+\nu+1}$, provided $\mu > 0$, $\nu > 0$.*

PROOF. We can suppose without loss of generality that the sum of Σa_n as well as of Σb_n is zero, and we have,*

$$\int_0^1 J_\mu(zt)\, J_\nu(Z\sqrt{1-t^2})\, t^{\mu+1}(1-t^2)^{\nu/2}dt = z^\mu\, Z^\nu \frac{J_{\mu+\nu+1}(\sqrt{z^2+Z^2})}{(Z^2+z^2)^{\frac{1}{2}(\mu+\nu+1)}}.$$

That is,

$$\int_0^1 \frac{J_\mu(mt)}{(mt)^\mu} \cdot \frac{J_\nu(n\sqrt{1-t^2})}{(n\sqrt{1-t^2})^\nu} \cdot t^{2\mu+1}(\sqrt{1-t^2})^{2\nu}dt = \frac{J_{\mu+\nu+1}(\sqrt{m^2+n^2})}{(m^2+n^2)^{\frac{1}{2}(\mu+\nu+1)}}$$

or,

$$\int_0^1 \alpha_\mu(mt)\, \alpha_\nu(n\sqrt{1-t^2})\, t^{2\mu+1}(\sqrt{1-t^2})^{2\nu}\, dt = \frac{\Gamma(\mu+1)\,\Gamma(\nu+1)}{2\Gamma(\mu+\nu+2)}\, \alpha_{\mu+\nu+1}(\sqrt{m^2+n^2}),$$

which leads to

$$\int_0^1 \alpha_\mu(mxt)\, \alpha_\nu(nx\sqrt{1-t^2})\, t^{2\mu+1}(1-t^2)^\nu\, dt = \frac{\Gamma(\mu+1)\,\Gamma(\nu+1)}{2\Gamma(\mu+\nu+2)}\, \alpha_{\mu+\nu+1}(x\sqrt{m^2+n^2}).$$

Multiplying the left side by Σa_m and Σb_n, and observing that $\Sigma a_m \alpha_\mu(mxt) = \phi_\mu(xt)$, and $\Sigma b_n \alpha_\nu(nxt) = \psi_\nu(xt)$, we obtain

$$\int_0^1 \phi_\mu(xt)\, \psi_\nu(x\sqrt{1-t^2})\, t^{2\mu+1}(1-t^2)^\nu\, dt = \frac{\Gamma(\mu+1)\,\Gamma(\nu+1)}{2\Gamma(\mu+\nu+2)} \Sigma c_r\, \alpha_{\mu+\nu+1}(x\sqrt{m^2+n^2}). \qquad (3.1)$$

*G. N. Watson, *Theory of Bessel Functions*, p. 376; use $\sin\theta = t$.

But,

$$I=\int_0^1 \phi_\mu(xt)\ \psi_\nu(x\sqrt{1-t^2})\ t^{2\mu+1}(1-t^2)^\nu\ dt$$

$$=\frac{1}{x^{2\mu+2\nu+2}}\int_0^x \phi_\mu(y)\ \psi_\nu(\sqrt{x^2-y^2})\ y^{2\mu+1}(x^2-y^2)^\nu\ dy$$

$$<\varepsilon\cdot\frac{1}{x^{2\mu+2\nu+2}}\int_0^x y^{2\mu+1}(x^2-y^2)^\nu\ dy,\ \text{ for } \varepsilon>0$$

and x sufficiently small

$$<\varepsilon\cdot\tfrac{1}{2}\,B(\mu+1,\ \nu+1). \tag{3.2}$$

Hence $\Sigma c_r\ \alpha_{\mu+\nu+1}\ (x\sqrt{m^2+n^2})$ converges for $x>0$, and tends to zero with x. Therefore, Σc_r is summable $\mathcal{J}_{\mu+\nu+1}$.

4. A series Σa_m is said to be absolutely summable* $\mathcal{J}_\mu$ or summable $|\mathcal{J}_\mu|$, if $\sum_{n=0}^{\infty} a_m\alpha_\mu(mt)$ converges for $t>0$, and if $\phi_\mu(t)=\Sigma a_m\alpha_\mu(mt)$ is of bounded variation in some finite interval enclosing the origin.

We now proceed to sum the Bessel-product of two series, one of which is simply summable $\mathcal{J}_\mu$ and the other is absolutely summable $\mathcal{J}_\nu$.

THEOREM 2. *If Σa_m is summable $|\mathcal{J}_\mu|$ and Σb_n is summable $\mathcal{J}_\nu$, the Bessel-product Σc_r is summable $\mathcal{J}_{\mu+\nu}$, when $\mu, \nu>0$.*

PROOF. Let $\Sigma c_r\alpha_{\mu+\nu}(rt)=\chi_{\mu+\nu}(t)$. Then, from (3.1),

$$\chi_{\mu+\nu}(t)=k_1\int_0^1 \psi_\nu(x\sqrt{1-t^2})\ t^{2\mu-1}(1-t^2)^\nu\ \phi_{\mu-1}(xt)\ dt, \tag{4.1}$$

where k_1 is a constant. Since,

$$\phi_{\mu-1}(xt)=\phi_\mu(xt)-k_2\ xt\ \phi_\mu'(xt)\ \dagger, \tag{4.2}$$

$\chi_{\mu+\nu}(t)=I_1+I_2$, where

$$I_1=k_1\int_0^1 \psi_\nu(x\sqrt{1-t^2})\ t^{2\mu-1}(1-t^2)^\nu\ \phi_\mu(xt)\ dt$$

$$I_2=k_2\int_0^1 \psi_\nu(x\sqrt{1-t^2})\ t^{2\mu}(1-t^2)^\nu\ \phi_\mu'(xt)\ dt.$$

* K. Chandrasekharan, The Absolute Bessel summability of series : *Bull. Cal. Math. Soc.* 34 (1942), 187-96.

† K. Chandrasekharan: The Bessel Summation of Series, *loc. cit.*

Now $I_1 = \chi_{\mu+\nu+1}(t) \to 0$ with t by Theorem 1.

$$|I_2| \leqslant k_3 \int_0^1 |\psi_\nu(x\sqrt{1-t^2})\, \phi_\mu'(xt)|\, xdt$$

$$\leqslant \epsilon k_3 \int_0^1 |\phi_\mu'(xt)|\, xdt, \text{ since } \Sigma b_n \text{ is summable } J_\nu.$$

$$\leqslant \epsilon k_4, \text{ since } \Sigma a_m \text{ is summable } |J_\mu|. \tag{4.3}$$

Hence $\chi_{\mu+\nu}(t)$ which is finite for $t > 0$, tends to zero with t, which means that Σc_r is summable $J_{\mu+\nu}$.

THEOREM 3. *If Σa_m is summable $|J_\mu|$ and Σb_n is summable $|J_\nu|$, then Σc_r is summable $|J_{\mu+\nu+1}|$, provided $\mu, \nu > 0$.*

PROOF.

$$\chi_{\mu+\nu+1}(x) = k_1 \int_0^1 \phi_\mu(xt)\, \psi_\nu(x\sqrt{1-t^2})\, t^{2\mu+1}(1-t^2)^\nu dt.$$

Hence $$\frac{d}{dx}[\chi(x)] = \theta_1(x) + \theta_2(x), \text{ where,}$$

$$\theta_1(x) = \int_0^1 t^{2\mu+2}(1-t^2)^\nu [\phi_\mu'(xt)\, \psi_\nu(x\sqrt{1-t^2})]dt,$$

$$\theta_2(x) = \int_0^1 t^{2\mu+1}(1-t^2)^{\nu+\frac{1}{2}}\, \phi_\mu(xt)\, \psi_\nu'(x\sqrt{1-t^2})\, dt.$$

Now, since $0 \leqslant x\sqrt{1-t^2} \leqslant x \leqslant 1$,

$$|\theta_1(x)| \leqslant k_2 \int_0^1 t^{2\mu+2}(1-t^2)^\nu |\phi_\mu'(xt)|\, dt,$$

so that

$$\int_0^1 |\theta_1(x)|\, dx \leqslant k_2 \int_0^1 dx \int_0^1 t^{2\mu+2}(1-t^2)^\nu |\phi_\mu'(xt)|\, dt$$

$$= k_2 \int_0^1 (1-t^2)^\nu t^{2\mu+2} dt \int_0^1 |\phi_\mu'(xt)|\, dx. \tag{4.4}$$

Now, $$\int_0^1 |\phi_\mu'(xt)|\, dx = \frac{1}{t}\int_0^t |\phi_\mu'(y)|\, dy = \frac{\Phi(t)}{t}, \text{ say.} \tag{4.5}$$

Hence, $$\int_0^1 |\theta_1(x)|\, dx \leqslant k_2 \int_0^1 (1-t^2)^\nu t^{2\mu+1}\, \Phi(t)\, dt$$

$$\leqslant k_3 \int_0^1 (1-t^2)^\nu t^{2\mu+1} dt, \text{ since } \Phi(t) \text{ is finite}$$

$$= k_3 \int_0^1 (1-x)^\nu x^\mu dx < \infty.$$

If we substitute $\sqrt{1-t^2} = \tau$ in θ_2, we obtain the same expression as in θ_1 with μ, ν and θ, ψ interchanged, so that we can similarly prove that

$$\int_0^1 | \theta_2(x) | \, dx < \infty .$$

Thus, $\left| \frac{d}{dx} \chi(x) \right|$ is integrable in $(0, 1)$, which proves the theorem.

My thanks are due to Dr. S. Minakshisundaram for his suggestions.

STEADY UNIPLANAR SUPERPOSABLE FLUID MOTIONS

BY

RAM BALLABH, *Lucknow University*

[Received 27 October 1942.]

The object of this paper* is to investigate the types of steady rotational and irrotational uniplanar motions in an inviscid homogeneous incompressible fluid which are mutually superposable. First, a general discussion of steady uniplanar motions superposable on an irrotational flow is given, and then the particular case of two-dimensional motions is considered.

1. Let $(u_1, v_1, 0)$ be irrotational, $(u_2, v_2, 0)$ rotational, and let ξ_2, η_2, ζ_2 be the vorticity components of the rotational motion, the planes of motion being parallel to $z=0$.

Since $(u_1, v_1, 0)$ is irrotational, u_1, v_1 must be independent of z and the conditions of superposability of $(u_1, v_1, 0)$ on $(u_2, v_2, 0)$ can be written as†

$$u_1\frac{\partial \zeta_2}{\partial x}+v_1\frac{\partial \zeta_2}{\partial y}=0 \tag{1}$$

$$u_1\frac{\partial \xi_2}{\partial x}+v_1\frac{\partial \xi_2}{\partial y}=\xi_2\frac{\partial u_1}{\partial x}+\eta_2\frac{\partial v_1}{\partial x} \tag{2}$$

$$u_1\frac{\partial \eta_2}{\partial x}+v_1\frac{\partial \eta_2}{\partial y}=\xi_2\frac{\partial u_1}{\partial y}+\eta_2\frac{\partial v_1}{\partial y}. \tag{3}$$

* I am grateful to Prof. J. A. Strang and Dr. R. S. Varma for their assistance in the preparation of this paper.

† See 'Superposable Fluid Motions', *Proc. Benares Math. Soc.* (2) 2 (1940), 71, (1.3).

The conditions of integrability of the equations of motion of $(u_2, v_2, 0)$ are*

$$u_2 \frac{\partial \zeta_2}{\partial x} + v_2 \frac{\partial \zeta_2}{\partial y} = 0 \qquad (4)$$

$$u_2 \frac{\partial \xi_2}{\partial x} + v_2 \frac{\partial \xi_2}{\partial y} = \xi_2 \frac{\partial u_2}{\partial x} + \eta_2 \frac{\partial v_2}{\partial x} \qquad (5)$$

$$u_2 \frac{\partial \eta_2}{\partial x} + v_2 \frac{\partial \eta_2}{\partial y} = \xi_2 \frac{\partial u_2}{\partial y} + \eta_2 \frac{\partial v_2}{\partial y}. \qquad (6)$$

1.1. If $\partial \zeta_2/\partial x$ and $\partial \zeta_2/\partial y$ are not zero, then from (1) and (4)

$$\frac{u_1}{v_1} = \frac{u_2}{v_2} = f(x, y). \qquad (7)$$

Also
$$\eta_2 = \frac{\partial u_2}{\partial z} = f \frac{\partial v_2}{\partial z} = -f \xi_2.$$

From (2) and (7), we get, since $v_1 \neq 0$

$$f \frac{\partial \xi_2}{\partial x} + \frac{\partial \xi_2}{\partial y} = \xi_2 \frac{\partial f}{\partial x}$$

or
$$\frac{\partial}{\partial z}\left(f \frac{\partial v_2}{\partial x} + \frac{\partial v_2}{\partial y} - v_2 \frac{\partial f}{\partial x} \right) = 0$$

or
$$\frac{\partial}{\partial z}\left\{ \frac{\partial}{\partial x}(f v_2) + \frac{\partial v_2}{\partial y} - 2 v_2 \frac{\partial f}{\partial x} \right\} = 0.$$

From (7) and the continuity condition of $(u_2, v_2, 0)$, we get

$$v_2 \frac{\partial f}{\partial x} = F(x, y),$$

i.e. v_2 is independent of z. u_2 is therefore also independent of z, and $\xi_2 = 0$, $\eta_2 = 0$. The problem becomes two-dimensional which will be discussed in section 2.

1.2. If $\partial \zeta_2/\partial x \neq 0$ and $\partial \zeta_2/\partial y = 0$, then from (1) and (4), $u_1 = 0$, $u_2 = 0$ and therefore $\eta_2 = 0$, $\partial v_1/\partial y = \partial v_2/\partial y = 0$, the last two following from the continuity conditions of the motions. Since $(u_1, v_1, 0)$ is irrotational, it follows that $\partial v_1/\partial x = 0$. Hence $v_1 = V$, V being a constant.

129992
-600 008.

* Cf. Lamb, *Hydrodynamics* (1932), p. 578, (8) and use the fact that $(u_2, v_2, 0)$ is steady.

(3) and (6) are identically satisfied, and from (2) and (5) we have

$$v_1\frac{\partial \xi_2}{\partial y}=0, \quad v_2\frac{\partial \xi_2}{\partial y}=0,$$

i.e. $\xi_2=f(x, z)$ and $v_2=F(x, z)$ such that $\partial F/\partial z+f=0$.

If on the other hand we had assumed $\partial \zeta_2/\partial y\neq 0$, $\partial\zeta_2/\partial x=0$, we would have obtained

$$u_1=U, \; v_1=0$$

$$u_2=G(y, z), \; v_2=0, \; \xi_2=0 \text{ and } \eta_2=\frac{\partial G}{\partial z}.$$

If, therefore, the vorticity of the steady rotational motion about the z-axis is independent of x or of y, then for superposition the irrotational motion must be constant streaming parallel to the x- or to the y-axis, and the stream lines of the rotational motion must be the straight lines y or $x=c_1$; $z=c_2$; c_1, c_2 being constants.

1.3. If $\partial\zeta_2/\partial x=\partial\zeta_2/\partial y=0$, then integrating the equations of superposability* after substituting the values of ξ_2, η_2 in terms of u_2, v_2 we get

$$u_1u_2+v_1v_2=P+\psi_1\zeta_2+X, \tag{8}$$

where P is a function of x and y, X a function of z, and ψ_1 the stream function of $(u_1, v_1, 0)$.

The condition of integrability of the equations of motion of $(u_2, v_2, 0)$ which is also the condition of self-superposability of $(u_2, v_2, 0)$ can be written as†

$$\tfrac{1}{2}(u_2^2+v_2^2)=Q+\psi_2\zeta_2+\Upsilon,$$

where Q is a function of x and y, Υ a function of z, and ψ_2 the stream function of $(u_2, v_2, 0)$.

As an example we may take $\psi_2=\frac{1}{2}y^2f(z)$. ψ_1 is then $a+by-cxy$, a, b, c being constants, provided f is not a constant. The stream lines of the irrotational motion in

* Equations (1.4), 'Superposable Fluid Motions' *loc. cit.*

† Integrate equations (1.4′) of the above paper, proceeding as for the deduction of (8).

this example are rectangular hyperbolæ and those of the rotational are the straight lines $y=$ constant, $z=$ constant.

2. In this section we are concerned with the two-dimensional case. Dropping the suffix of u_1, v_1, the condition of superposability is

$$u\frac{\partial \zeta_2}{\partial x}+v\frac{\partial \zeta_2}{\partial y}=0, \tag{2.1}$$

and the condition of integrability of the equations of motion of (u_2, v_2) is

$$u_2\frac{\partial \zeta_2}{\partial x}+v_2\frac{\partial \zeta_2}{\partial y}=0. \tag{2.2}$$

Hence in two-dimensional flow a motion with constant vorticity can be superimposed upon any irrotational motion.

If the vorticity is not constant, we get from (1) and (2), $\psi_2=h(\psi)$ where the suffix of ψ_1 has been dropped. Now

$$\zeta_2=\nabla_1^2\psi_2=h''(\psi)\left\{\psi_x^2+\psi_y^2\right\}.$$

Combining this with (2.1), viz., with $\zeta_2=g(\psi)$, we have $\psi_x^2+\psi_y^2$ either a constant or a function of ψ.

If $\psi_x^2+\psi_y^2=$ constant, ψ_x and ψ_y must be constants because the motion is irrotational. Motions with constant streaming are thus superposable on a steady rotational motion with variable vorticity.

If $\psi_x^2+\psi_y^2$ is a function of ψ, i.e., if $q=f(\psi)$, we consider*

$$q_x^2+q_y^2=q\nabla_1^2 q$$

so that $$\left\{f'(\psi)\right\}^2\left\{\psi_x^2+\psi_y^2\right\}=f(\psi)\,f''(\psi)\left\{\psi_x^2+\psi_y^2\right\}$$

or $$\frac{f'(\psi)}{f(\psi)}=\frac{f''(\psi)}{f'(\psi)}$$

or $$f(\psi)=be^{a\psi},\ a,\ b \text{ being constants.}$$

* Milne-Thomson, *Theoretical Hydrodynamics* (1938), p. 112.

ψ is therefore given by

$$\psi_x^2+\psi_y^2=b^2e^{2a\psi} \tag{2.3}$$

$$\psi_{xx}+\psi_{yy}=0. \tag{2.4}$$

To solve these, putting $e^{-a\psi}=\chi$ we get

$$\chi_x^2+\chi_y^2=a^2b^2=c^2 \tag{2.5}$$

$$\chi(\chi_{xx}+\chi_{yy})=c^2. \tag{2.6}$$

Differentiating (2.5), we have

$$\chi_x\chi_{xx}+\chi_y\chi_{xy}=0 \tag{2.7}$$

$$\chi_x\chi_{xy}+\chi_y\chi_{yy}=0, \tag{2.8}$$

from which, $\chi_x^2\chi_{xx}-\chi_y^2\chi_{yy}=0$

which with (2.6) and (2.5) gives

$$\chi\chi_{xx}=\chi_y^2 \tag{2.9}$$

$$\chi\chi_{yy}=\chi_x^2. \tag{2.10}$$

Multiplying (2.7) by χ_y, (2.8) by χ_x, adding and using (2.5) and (2.6),

$$\chi_x\chi_y+\chi\chi_{xy}=0$$

i.e.,

$$\frac{\partial}{\partial x}(\chi\chi_y)=0 \tag{2.11}$$

and

$$\frac{\partial}{\partial y}(\chi\chi_x)=0. \tag{2.12}$$

From (2.9),

$$\chi\chi_{xx}+\chi_x^2=\chi_y^2+\chi_x^2=c^2$$

or

$$\frac{\partial}{\partial x}(\chi\chi_x)=c^2. \tag{2.13}$$

Similarly from (2.10),

$$\frac{\partial}{\partial y}(\chi\chi_y)=c^2. \tag{2.14}$$

From these equations we get,

$$\chi\chi_x=c(cx+A),$$

$$\chi\chi_y=c(cy+B),$$ A and B being constants.

Hence from (2.5), $\chi^2=(cx+A)^2+(cy+B)^2$,

and this satisfies (2.6). We obtain also

$$\psi=-a^{-1}\log\sqrt{(cx+A)^2+(cy+B)^2}.$$

The stream lines of the irrotational motion are therefore circles which are also the stream lines of the rotational motion. Thus for the two-dimensional case,

(1) Any irrotational motion can be superimposed upon a rotational motion with constant vorticity.

(2) If the vorticity of the steady rotational motion is not constant in space, then its stream function and vorticity must depend solely on the stream function of the irrotational motion, which has the form

(i) $\alpha x+\beta y+\gamma$, α, β, γ being constants

or (ii) $-a^{-1}\log\sqrt{(cx+A)^2+(cy+B)^2}$.

THE CHARACTERS OF THE CLASSES $(n-k, k)$ OF THE SYMMETRIC GROUP OF DEGREE n.*

BY

T. VENKATARAYUDU, *Andhra University*.

[Received 8 January 1943.]

INTRODUCTION. D. E. Littlewood† showed that the quickest method of calculating the characters of the symmetric groups is by the use of the recurrence relations between the *S*-functions. He has given the tables of characters for the symmetric groups up to degree 10. Later M. Zia-Ud-Din‡ published tables for the symmetric groups of degrees 11, 12, 13. Any general result relating to the characters is of immense help in obtaining these tables. The aim of the present paper is to derive formulae for the characters of the classes $(n-k, k)$ in the irreducible representations of the symmetric group of degree n. It may be mentioned that the recurrence relations given by Littlewood may be used to obtain the characters of all the classes except the class (n). Hitherto after getting the characters of all the other classes, the characters of this class are being obtained with the help of the orthogonality relations between the group characters. Here we give a direct formula for the characters of this class.

GRAPHS:—A graph consisting of nodes arranged in rows and columns is said to be regular if in any row which contains, say, r nodes, these occur in the first r

* My thanks are due to Dr. R. Vaidyanathaswamy for his constant help and encouragement.

† *The theory of group characters and matrix representations of groups*, 1940.

‡ *Proc. Lond. Math. Soc.* 1935, 39, 200 and 1937, 42, 340.

columns, and in any column which contains, say, r nodes these occur in the first r rows. If to a regular graph one node is added so that the resulting graph is also regular, this is called a regular application of a node. The addition of r nodes to a regular graph is called a regular application of r nodes if the nodes are added to any row until they are exhausted, or until the number of nodes in this row exceeds the number of nodes in the preceding row by one, the nodes then being added to the preceding row according to the same rule, and so on until the r nodes are exhausted, provided the final graph obtained is regular. If the number of rows involved is even, it is called a negative application, if odd a positive application. The following two* theorems are most useful in finding the character of any class in an irreducible representation of a symmetric group corresponding to a partition of the degree of the group.

THEOREM 1. *If* $[\lambda] = [\lambda_1, \lambda_2, \ldots, \lambda_p]$ *is a partition of* n, *and* ρ *denotes the class of the symmetric group with cycles of orders* $a_1, a_2, \ldots$ *then* $\chi_\rho^{[\lambda]}$ *is obtained from the number of methods of building the graph of the partition* λ *by consecutive regular applications of* $a_1, a_2, \ldots$ *nodes, by subtracting the number of methods which contain an odd number of negative applications from the number of methods which contain an even number of negative applications.*

The following theorem is an immediate consequence.

THEOREM 2.

$$\chi_\rho^{[n]} = 1 \quad \textit{for every class.}$$
$$\chi_\rho^{[1^n]} = 1 \quad \textit{for a positive class,}$$
$$= -1 \quad \textit{for a negative class.}$$

From Theorems 1 and 2, we have

THEOREM 3. *The characters of the class* (n) *in the irreducible representations of the symmetric group of degree* n *corresponding to the partitions*

* See Littlewood, *loc. cit.* p. 70.

$[n]$; $[1^n]$; $[p, 1^q]$ *where* $p+q=n$ *are respectively* 1; $(-1)^{n-1}$; $(-1)^q$. *The characters for the same class in the other irreducible representations are all zero.*

PROOF. Cases 1 and 2 follow from Theorem 2.

Partition $[p, 1^q]$. There is only one method of building the graph of this partition by regular application of n nodes, and the number of rows involved is $q+1$. Hence

$$\chi_{(n)}^{[p, 1^q]} = (-1)^q.$$

For the partitions other than those given above, it is not possible to build a graph by regular applications of n nodes and therefore $\chi_{(n)}^{[\lambda]} = 0$.

From Theorems 1, 2, 3, we have

THEOREM 4. *The characters of the class* $(n-1, 1)$ *in the irreducible representations corresponding to the partitions* $[n]$; $[1^n]$; $[p, 2, 1^{q-2}]$ *are respectively* 1, $(-1)^{n-2}$, $(-1)^{q-1}$. *The characters for the same class in the other irreducible representations are all zero.*

PROOF. To build the graph of a partition μ of n by regular applications of $n-1$, 1 nodes we first build the graph of the partition λ of $n-1$ by regular application of $n-1$ nodes and then complete the graph of the partition μ by a regular application of one more node. By Theorem 3 it is possible to build the graphs of the partitions $[n-1]$; $[1^{n-1}]$; $[p, 1^q]$, where $p+q=n-1$ only by the regular application of $n-1$ nodes. A regular application of one more node to the above graphs gives rise to the graphs of the partitions:

$[n]$; $[n-1, 1]$; $[1^n]$; $[2, 1^{n-2}]$; $[p+1, 1^q]$; $[p, 2, 1^{q-1}]$; $[p, 1^{q+1}]$ where $p+q=n-1$. Now by Theorem 2.

$$\chi_{(n-1, 1)}^{[n]} = 1.$$

There is only one method of building the graph of the partition $[1^n]$ by the regular applications of $n-1$, 1 nodes and it involves a negative application if n is odd and no negative application if n is even. Hence

$$\chi_{(n-1, 1)}^{[1^n]} = (-1)^{n-2}.$$

The graph of the partition $[n-1, 1]$ can be built by the regular application of $n-1$, 1 nodes in only two ways one of which contains one negative application and the other contains no negative application. Hence

$$\chi_{(n-1,\,1)}^{[n-1,\,1]} = 0.$$

Similarly

$$\chi_{(n-1,\,1)}^{[2,\,1^{n-2}]} = 0; \quad \chi_{(n-1,\,1)}^{[p+1,\,1^{q}]} = 0; \quad \chi_{(n-1,\,1)}^{[p,\,1^{q+1}]} = 0.$$

There is only one way of building the graph of the partition $[p, 2, 1^{q-1}]$ and this is by adding one node in the second row of the graph of the partition $[p, 1^q]$ of $n-1$. Hence

$$\chi_{(n-1,\,1)}^{[p,\,2,\,1^{q-1}]} = \chi_{(n-1)}^{[p,\,1^{q}]} = (-1)^q.$$

Since no graph can be built by the regular application of $n-1$, 1 nodes corresponding to the remaining partitions $[\lambda]$ of n, we have

$$\chi_{(n-1,\,1)}^{[\lambda]} = 0.$$

In a similar way we prove

THEOREM 5. *If ρ is the class (n_1, n_2), $n_2 \neq 0$ or 1*

(1) $\chi_{\rho}^{[n-r,\,1^{r}]} = (-1)^r$, $r = 0, 1, 2, \ldots, n_2 - 1$

(2) $\chi_{\rho}^{[n_1,\,1^{n_2}]} = 0$ *if* $n_1 \neq n_2$
$\quad = (-1)^{n_2-1}$ *if* $n_1 = n_2$

(3) $\chi_{\rho}^{[n_1-r+1,\,n_2-q+1,\,2^{r-1},\,1^{q-r}]} = (-1)^{q+r}$ *if* $n_1 \neq n_2$
$\quad = 2(-1)^{q+r}$ *if* $n_1 = n_2$

(4) $\chi_{\rho}^{[n_1-q,\,n_2-r+2,\,2^{r-1},\,1^{q-r}]} = (-1)^{q+r-1}$

(5) $\chi_{\rho}^{[n_2-r+1,\,n_1-q+1,\,2^{r-1},\,1^{q-r}]} = (-1)^{q+r}$ *if* $n_1 \neq n_2$
$\quad = 2(-1)^{q+r}$ *if* $n_1 = n_2$

(6) $\chi_{\rho}^{[n_2+1,\,1^{n_1-1}]} = 0$ *if* $n_1 \neq n_2$
$\quad = (-1)^{n_1-1}$ *if* $n_1 = n_2$

(7) $\chi_{\rho}^{[n_2-r+1,\,1^{n_1+r-1}]} = (-1)^{n_1+r}$, $r = 1, 2, \ldots, n_2$.

The characters of the class (n_1, n_2) in all other representations are zero.

SOME INFINITE INTEGRALS INVOLVING WHITTAKER FUNCTIONS

BY

B. R. PASRICHA, *Isabella Thoburn College, Lucknow.*

[Received 15 December 1942]

1. The object of this paper is to evaluate

$$I=\int_0^\infty x^{\mu-1}\,e^{-(\alpha^2+\frac{1}{2})x}\,W_{k,\,\nu}(x)\,{}_rF_s\left(\begin{matrix}l_1,\,l_2,\ldots,\,l_r\\ m_1,\,m_2,\ldots,\,m_s\end{matrix};\,yx^p\right)dx,\qquad (1)$$

and consider some interesting particular cases.

We write the equivalent infinite series

$$\sum_{m=0}^{\infty}\frac{(l_1,\,m)\;(l_2,\,m)\ldots(l_r,\,m)}{(m_1,\,m)\;(m_2,\,m)\ldots(m_s,\,m)m!}\,y^p x^{pm}$$

for the generalized hypergeometric function ${}_rF_s$. Integrating term by term and evaluating with the help of Goldstein's* integral, we get

$$I=\sum_{m=0}^{\infty}\frac{(l_1,\,m)\;(l_2,\,m)\ldots(l_r,\,m)\;y^m\;\Gamma(a+pm)\;\Gamma(b+pm)}{(m_1,\,m)\;(m_2,\,m)\ldots(m_s,\,m)\;m!\;\Gamma(c+pm)}$$

$$\times\;{}_2F_1\left(\begin{matrix}a+pm,\;b+pm\\ c+pm\end{matrix};\,-\alpha^2\right)\qquad (2)$$

where $a=\mu+\nu+\frac{1}{2}$, $b=\mu-\nu+\frac{1}{2}$, and $c=\mu-k+1$.

On expanding ${}_2F_1$ and changing the order of summation, we can also write the result in the form

* *Proc. Lond. Math. Soc.* 34 (1932), 103-25.

$$I = \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\,\Gamma(b+n)(-\alpha^2)^n}{\Gamma(c+n)\,n!}$$

$$\times {}_{r+2p}F_{s+p}\left\{\begin{matrix} \frac{a+n}{p}, \frac{a+n+1}{p}, \ldots, \frac{a+n+p-1}{p}, \frac{b+n}{p}, \ldots, \frac{b+n+p-1}{p}, l_1, \ldots, l_r; \\ \frac{c+n}{p}, \frac{c+n+1}{p}, \ldots, \frac{c+n+p-1}{p}, m_1, m_2, \ldots, m_s; \end{matrix} p^p y\right\}. \qquad (3)$$

The above result is valid under the conditions, a and both > 0, $R(\alpha^2+1) > 0$, $|\alpha^2| < 1$

(i) for all values of y when $r+p \leqslant s$,

(ii) for $|y| < 1/p^p$ when $r+p = s+1$, and

(iii) when $r+p \geqslant s+2$ only if ${}_{r+2p}F_{s+p}$ is a terminating series.

The above result is important from several points of view and we will now indicate a few of these.

2. The general results reduce to very elegant forms when $\alpha = 0$ and when $\alpha^2 = -\frac{1}{2}$.

(i) When $\alpha = 0$ we have from (3) or from (2)

$$\int_0^\infty x^{\mu-1} e^{-\frac{1}{2}x}\, W_{k,\nu}(x)\, {}_rF_s\left(\begin{matrix} l_1, l_2, \ldots, l_r \\ m_1, m_2, \ldots, m_s \end{matrix}; yx^p\right) dx$$

$$= \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(c)}$$

$$\times {}_{r+2p}F_{s+p}\left\{\begin{matrix} \frac{a}{p}, \frac{a+1}{p}, \ldots, \frac{a+p-1}{p}, \frac{b}{p}, \ldots, \frac{b+p-1}{p}, l_1, \ldots, l_r \\ \frac{c}{p}, \frac{c+1}{p}, \ldots, \frac{c+p-1}{p}, m_1, \ldots, m_s \end{matrix}; p^p y\right\}. \qquad (4)$$

(ii) When $\alpha^2 = -\frac{1}{2}$ and k is zero, starting from (2) we have by the help of Gauss' summation formula,

$$\int_0^\infty x^{\mu-\frac{1}{2}}\, K_\nu(\tfrac{1}{2}x)\, {}_rF_s\left(\begin{matrix} l_1, l_2, \ldots, l_r; \\ m_1, m_2, \ldots, m_s; \end{matrix} yx^p\right) dx$$

$$= 2^{2\mu-1}\,\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{b}{2}\right)$$

$$\times\,{}_{2r+2p}F_{2s+1}\left\{\begin{matrix}\frac{a}{2p}, \frac{a+2}{2p}, \ldots, \frac{a+2p-2}{2p}, \frac{b}{2p}, \ldots, \frac{b+2p-2}{2p}, \frac{l_1}{2}, \frac{l_1+1}{2}, \ldots, \frac{l_r}{2}, \frac{l_r+1}{2}; \\ \frac{1}{2}, \frac{m_1}{2}, \frac{m_1+1}{2}, \frac{m_2}{2}, \frac{m_2+1}{2}, \ldots, \frac{m_s}{2}, \frac{m_s+1}{2}; \; 4^{2p+r-s-1}\, p^{2p}\, y^2\end{matrix}\right\}$$

$$+\frac{2^{2\mu+2p-1}\,\Gamma\left(\frac{a+p}{2}\right)\Gamma\left(\frac{b+p}{2}\right) l_1\, l_2 \ldots l_r}{m_1\, m_2 \ldots m_s}\, y$$

$$\times\,{}_{2r+2p}F_{2s+1}\left\{\begin{matrix}\frac{a}{2p}+\frac{1}{2}, \frac{a+2}{2p}+\frac{1}{2}, \ldots, \frac{a+2p-2}{2p}+\frac{1}{2}, \frac{b}{2p}+\frac{1}{2}, \ldots \frac{b+2p-2}{2p}+\frac{1}{2}, \frac{l_1+1}{2}, \frac{l_1+2}{2}, \ldots, \frac{l_r+1}{2}, \frac{l_r+2}{2}; \\ \frac{3}{2}, \frac{m_1+1}{2}, \frac{m_1+2}{2}, \ldots, \frac{m_s+1}{2}, \frac{m_s+2}{2}, \; 4^{2p+r-s-1}\, p^{2p}\, y^2\end{matrix}\right\}. \qquad (5)$$

When p is even we can also write the integral on the left as equal to

$$2^{2\mu-1}\,\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{b}{2}\right)$$

$$\times\,{}_{r+p}F_s\left\{\begin{matrix}\frac{a}{p}, \frac{a+2}{p}, \ldots, \frac{a+p-2}{p}, \frac{b}{p}, \ldots, \frac{b+p-2}{p}, l_1, \ldots, l_r; \\ m_1, m_2, \ldots m_s; \qquad (2p)^p y\end{matrix}\right\}. \qquad (6)$$

3. Further for $r=0$, (3) gives the integral, given below, involving the functions* defined by†

$$\mathcal{J}_{m_1,\ldots,m_s}(x) = \frac{x^{m_1+\ldots+m_s}}{(s+1)^{m_1+\ldots+m_s}\,\Gamma(m_1+1)\ldots\Gamma(m_s+1)}$$
$$\times\,{}_0F_s\left[m_1+1, \ldots, m_s+1; -(x/s+1)^{s+1}\right];$$

* The name Humbert Functions has been given to $\mathcal{J}_{m_1, m_2}(x)$ by Dr. R. S. Varma. I shall call the functions $\mathcal{J}_{m_1, m_2, \ldots, m_s}(x)$ with s denominators as Generalised Humbert Functions of order s.

† P. Humbert, *Atti. Pont. Acad. della Scienza*, 83 (1930), 146.

$$\int_0^\infty x^{\mu-(p\Sigma m_1/\overline{s+1})-1}\, e^{-(\alpha^2+\frac{1}{2})x}\, W_{k,\nu}(x)\, J_{m_1,\ldots,m_s}(yx^{p/(s+1)})\, dx$$

$$= \left(\frac{y}{s+1}\right)^{\Sigma m_1} \frac{1}{\Pi\,\Gamma(m_1+1)} \sum_{n=0}^{\infty} \Bigg[\frac{\Gamma(a+n)\,\Gamma(b+n)(-\alpha^2)^n}{n!\,\Gamma(c+n)} \cdot$$

$$\times\, {}_{2p}F_{s+p}\left\{ \begin{matrix} \frac{a+n}{p}, \frac{a+n+1}{p}, \ldots, \frac{a+n+p-1}{p}, \frac{b+n}{p}, \ldots, \frac{b+n+p-1}{p}; \\ \frac{c+n}{p}, \frac{c+n+1}{p}, \ldots \frac{c+n+p-1}{p}, m_1, m_2, \ldots, m_s; \end{matrix} \right.$$

$$\left. - \left(\frac{y}{s+1}\right)^{s+1} p^p \right\} \Bigg]. \qquad (7)$$

The case when $p = s+1$ is interesting,

$$\int_0^\infty x^{\mu-\Sigma m_1 -1}\, e^{-(\alpha^2+\frac{1}{2})x}\, W_{k,\nu}(x)\, J_{m_1,\ldots,m_s}(xy)\, dx \ldots$$

$$= \left(\frac{y}{s+1}\right)^{\Sigma m_1} \frac{1}{\Pi\,\Gamma(m_1+1)} \sum_{n=0}^{\infty} \Bigg[\frac{\Gamma(a+n)\,\Gamma(b+n)\,(-\alpha^2)^n}{n!\,\Gamma(c+n)}$$

$$\times\, {}_{2s+2}F_{2s+1}\left\{ \begin{matrix} \frac{a+n}{s+1}, \frac{a+n+1}{s+1}, \ldots, \frac{a+n+s}{s+1}, \frac{b+n}{s+1}, \ldots, \frac{b+n+s}{s+1}; \\ \frac{c+n}{s+1}, \frac{c+n+1}{s+1}, \ldots, \frac{c+n+s}{s+1}, m_1, m_2, \ldots, m_s; \end{matrix} \; - y^{s+1}) \right\} \qquad (8)$$

provided that $|y| < 1$.

4. (i) By a slight change of variable and taking $\mu = \lambda + \frac{1}{2}$, $r = 0$, $s = 3$ and $p = 2$, (6) gives us an integral due to C. S. Meijer.*

(ii) By taking $r = s = 1$ and $p = 1$, the relation (4) gives a result of R. S. Varma.†

(iii) Another integral due to R. S. Varma† can be obtained from (4) by taking $r = 2$, $p = 1$ and $s = 2$.

* *Q. Jour. Math.* (Oxford) 6 (1935), 246.

† *Jour. Indian Math. Soc.* 25 (1938), 25-33

(iv) An integral due to (Miss) S. Sinha* is included in the relation (4) for $r = s = 2$ and $p = 1$.

(v). Integrals I_1, I_2 deduced by the author in a paper to appear in *Proc. Benares Math. Soc.* 4 (1942) can similarly be obtained.

5. Finally it may be noted that integral expressions for generalized Humbert functions are also given by our results. These expressions are, however, not elegant and so have not been given.

I am grateful to Dr. R. S. Varma for his help in the preparation of this note.

* *Jour. Benares Hindu University*, 7 (1942), 1-17.

ON THE PROJECTIVE DEFORMATION OF A FAMILY OF ELEMENTS OF CONTACT

BY

HSIEN-CHUNG WANG*

(*Tsing-Hua University, now at Kunming, China*)

[Received 25 May, 1943]

The problem of projective deformation of surfaces has been the object of very extensive researches.† It is the aim of the present paper to study the same problem for a two-parameter family of elements of contact of Lie. In order for it to be different from the surface, we assume that the planes are not tangent to the surface S described by the origins of elements of contact. Two such families are called deformable,‡ if there can be established a correspondence between their elements of contact such that for every pair of corresponding elements there exists a projective transformation carrying the neighbourhood of the second order of the point and of the plane of the one to that of the other. Evidently, a necessary condition for a two-parameter family of elements of contact to be deformable is that the surface S is deformable. In what follows we shall prove that if S is a non-ruled deformable surface, then, corresponding to every conjugate net of deformation§ of S, there can be attached to every point

* The author wishes to express his thanks to Professor Shiing-shen Chern for his valuable suggestions.

† Cf., in particular, G. Fubini, Applicabilita proiettiva di due superficie, *Rendiconti del Circolo matem. di Palermo*, 41 (1916), 135-62, and E. Cartan, Sur la deformation projective des surfaces, *Ann. Ec. Norm. sup.*, (3), 37 (1920), 259-356. The latter paper will be cited in the following as Cartan, *Surfaces*.

‡ E. Cartan, Sur le probleme general de la deformation, *C. R. du Congres Int. des Math. de Strasbourg en* 1920; Toulouse (1921), 397-406.

§ Cartan, *Surfaces*.

A of S one and only one element of contact with A as origin such that the resulting family of elements of contact is deformable.

Following Cartan* we attach to every point A of S a Frenet frame $AA_1A_2A_3$ such that the equations of infinitesimal displacement of the family of frames $AA_1A_2A_3$ are*

$$\left.\begin{aligned} dA &= -\tfrac{3}{2}(\alpha\omega^1+\beta\omega^2)A+\omega^1A_1+\omega^2A_2, \\ dA_1 &= (\lambda\omega^1+\mu\omega^2)A+\tfrac{1}{2}(\alpha\omega^1-\beta\omega^2)A_1+\omega^1A_2+\omega^2A_3, \\ dA_2 &= (\nu\omega^1+\rho\omega^2)A+\omega^2A_1+\tfrac{1}{2}(-\alpha\omega^1+\beta\omega^2)A_2+\omega^1A_3, \\ dA_3 &= (\rho\omega^1+\lambda\omega^2)A+(\nu\omega^1+\rho\omega^2)A_1+(\lambda\omega^1+\mu\omega^2)A_2 \\ &\qquad +\tfrac{3}{2}(\alpha\omega^1+\beta\omega^2)A_3, \end{aligned}\right\} \tag{1}$$

where α, β, λ, μ, ν, ρ are the fundamental invariants of the surface. These invariants satisfy the following set of integrability conditions:

$$\left.\begin{aligned} &[\omega^1\{d\alpha+(1-\alpha\beta-\tfrac{4}{3}\mu-\tfrac{2}{3}\nu)\,\omega^2\}]=0, \\ &[\omega^2\{d\beta+(1-\alpha\beta-\tfrac{2}{3}\mu-\tfrac{4}{3}\nu)\,\omega^1\}]=0, \\ &[\omega^1d\nu]+[\omega^2d\rho]+(2\alpha\rho-3\beta\nu)[\omega^1\omega^2]=0, \\ &[\omega^1d\lambda]+[\omega^2d\mu]+(3\alpha\mu-2\beta\lambda)[\omega^1\omega^2]=0, \\ &[\omega^1d\rho]+[\omega^2d\lambda]+4(\alpha\lambda-\beta\rho)[\omega^1\omega^2]=0. \end{aligned}\right\} \tag{2}$$

Let $\bar{\omega}_\alpha^\beta$ $(\alpha, \beta=0, 1, 2, 3)$ be the relative components of a second family of frames $\bar{A}\,\bar{A}_1\,\bar{A}_2\,\bar{A}_3$ whose origin $\bar{A}$ describes a surface $\bar{S}$. Then the most general surface $\bar{S}$ deformable to S is defined by the following Pfaffian system†

$$\left.\begin{aligned} &\bar{\omega}^1=\omega^1,\ \bar{\omega}^2=\omega^2,\ \bar{\omega}_1^2=\omega^1,\ \bar{\omega}_2^1=\omega^2,\ \bar{\omega}^3=0, \\ &\bar{\omega}_1^3=\omega^2,\ \bar{\omega}_2^3=\omega^1,\ \bar{\omega}_3^1-\bar{\omega}_2^0=k\omega^1,\ \bar{\omega}_3^2-\bar{\omega}_1^0=k\omega^2, \\ &\bar{\omega}_1^0=\omega_1^0-\tfrac{1}{2}k\omega^2+u\omega^1,\ \bar{\omega}_2^0=\omega_2^0-\tfrac{1}{2}k\omega^1+v\omega^2, \\ &\bar{\omega}_0^0-\omega_0^0=\bar{\omega}_1^1-\omega_1^1=\bar{\omega}_2^2-\omega_2^2=\bar{\omega}_3^3-\omega_3^3, \end{aligned}\right\} \tag{3}$$

* Cartan, *Surfaces*.

† G. Fubini et E. Cech, *Geometrie projective differentielle des surfaces*, Paris (1931), p. 220.

where k, u, v are three new functions in the parameters of the surface and where ω_0^0, ω_1^1, ω_2^2, ω_3^3 stand for the Pfaffian forms in the principal diagonal of the matrix defined by (1).

We now introduce plane co-ordinates by putting*

$$a=[A_1A_2A_3],\ a_1=-[AA_2A_3],\ a_2=[AA_1A_3],\ a_3=-[AA_1A_2]. \qquad (4)$$

Then, if A, A_1, A_2, A_3 satisfy the equations

$$\left.\begin{aligned} dA &= \omega_0^0A+\omega^1A_1+\omega^2A_2+\omega^3A_3,\\ dA_1 &= \omega_1^0A+\omega_1^1A_1+\omega_1^2A_2+\omega_1^3A_3,\\ dA_2 &= \omega_2^0A+\omega_2^1A_1+\omega_2^2A_2+\omega_2^3A_3,\\ dA_3 &= \omega_3^0A+\omega_3^1A_1+\omega_3^2A_2+\omega_3^3A_3, \end{aligned}\right\} \qquad (5)$$

we have

$$\left.\begin{aligned} da &= -\omega_0^0a-\omega_1^0a_1-\omega_2^0a_2-\omega_3^0a_3,\\ da_1 &= -\omega^1a-\omega_1^1a_1-\omega_2^1a_2-\omega_3^1a_3,\\ da_2 &= -\omega^2a-\omega_1^2a_1-\omega_2^2a_2-\omega_3^2a_3,\\ da_3 &= -\omega^3a-\omega_1^3a_1-\omega_2^3a_2-\omega_3^3a_3. \end{aligned}\right\} \qquad (6)$$

A plane through A is defined by

$$\pi=h_1a_1+h_2a_2+h_3a_3. \qquad (7)$$

As A varies on S, the elements of contact with A as origin and π as plane form a two-parameter family of elements of contact Σ. Similarly, we can attach a plane $\bar{\pi}$ through $\bar{A}$ to every point $\bar{A}$ of $\bar{S}$. In order that the resulting family of elements of contact $\bar{\Sigma}$ be deformable to Σ, it is first of all necessary that

$$\bar{\pi}=h_1\bar{a}_1+h_2\bar{a}_2+h_3\bar{a}_3, \qquad (8)$$

where $\bar{a}_\alpha(\alpha=0, 1, 2, 3)$ are formed from A by formulae analogous to (4). The necessary and sufficient conditions for Σ and $\bar{\Sigma}$ to be deformable are then, up to and including the infinitesimals of the second order, that the

* Fubini and Cech: *Geometrie projective differentielle des surfaces* Paris, (1931), p. 220.

co-ordinates of $d\pi$, $d^2\pi$ referred to $aa_1a_2a_3$ are respectively equal to those of $d\bar{\pi}+\rho_1\bar{\pi}$, $d^2\bar{\pi}+2\rho_1 d\bar{\pi}+2\rho_2\bar{\pi}$ referred to $\bar{a}\bar{a}_1\bar{a}_2\bar{a}_3$ where ρ_1 is an infinitesimal of the first order and ρ_2 of the second order. Now we find from (7) and (6),

$$d\pi = -(h_1\omega^1+h_2\omega^2+h_3\omega^3)a+(dh_1-h_1\omega_1^1-h_2\omega_1^2-h_3\omega_1^3)a_1 \\ +(dh_2-h_1\omega_2^1-h_2\omega_2^2-h_3\omega_2^3)a_2+(dh_3-h_1\omega_3^1-h_2\omega_3^2-h_3\omega_3^3)a_3,$$

and, if we write

$$d\pi = \vartheta_0 a+\vartheta_1 a_1+\vartheta_2 a_2+\vartheta_3 a_3,$$

we have the relation

$$\begin{aligned} d^2\pi = {} & (d\vartheta_0-\vartheta_0\omega_0^0-\vartheta_1\omega^1-\vartheta_2\omega^2-\vartheta_3\omega^3)a \\ & +(d\vartheta_1-\vartheta_0\omega_1^0-\vartheta_1\omega_1^1-\vartheta_2\omega_1^2-\vartheta_3\omega_1^3)a_1 \\ & +(d\vartheta_2-\vartheta_0\omega_2^0-\vartheta_1\omega_2^1-\vartheta_2\omega_2^2-\vartheta_3\omega_2^3)a_2 \\ & +(d\vartheta_3-\vartheta_0\omega_3^0-\vartheta_1\omega_3^1-\vartheta_2\omega_3^2-\vartheta_3\omega_3^3)a_3. \end{aligned}$$

From the necessary and sufficient conditions stated above, it follows that

$$h_1(\bar{\omega}_0^0-\omega_0^0)=h_1\rho_1, \quad h_2(\bar{\omega}_0^0-\omega_0^0)=h_2\rho_1,$$

$$h_1(\bar{\omega}_3^1-\omega_3^1)+h(\bar{\omega}_3^2-\omega_3^2)+h_3(\bar{\omega}_0^0-\omega_0^0)=h_3\rho_1.$$

Since, by assumption, the plane π does not coincide with the tangent plane a_3 of the surface S, h_1 and h_2 cannot vanish simultaneously and we find, from the above set of equations,

$$\rho_1=\bar{\omega}_0^0-\omega_0^0, \quad h_1(\bar{\omega}_3^1-\omega_3^1)+h_2(\bar{\omega}_3^2-\omega_3^2)=0.$$

Hence we obtain the following conditions for the deformability of Σ and $\bar{\Sigma}$,

$$\left.\begin{aligned} & h_1(\bar{\omega}_3^1-\omega_3^1)+h_2(\bar{\omega}_3^2-\omega_3^2)=0, \\ & \rho_2'h_1=\vartheta_0(\bar{\omega}_1^0-\omega_1^0),\ \rho_2'h_2=\vartheta_0(\bar{\omega}_2^0-\omega_2^0), \\ & \rho_2'h_3=\vartheta_0(\bar{\omega}_3^0-\omega_3^0)+\vartheta_1(\bar{\omega}_3^1-\omega_3^1)+\vartheta_2(\bar{\omega}_3^2-\omega_3^2), \end{aligned}\right\} \tag{9}$$

where ρ_2' is a quadratic differential form. If $\rho_2'=0$, we have, since $\vartheta_0\neq 0$,

$$\bar{\omega}_1^0-\omega_1^0=0, \quad \bar{\omega}_2^0-\omega_2^0=0,$$

and it is easy to deduce from (3) and (9) that Σ and $\bar{\Sigma}$ are projectively equivalent. We shall assume therefore that $\rho_2' \neq 0$. Then the last four equations of (3) may be written

$$\bar{\omega}_1^0 - \omega_1^0 = u\omega^1 - \tfrac{1}{2}k\omega^2, \quad \bar{\omega}_2^0 - \omega_2^0 = -\tfrac{1}{2}k\omega^1 + v\omega^2,$$
$$\bar{\omega}_3^1 - \omega_3^1 = \tfrac{1}{2}k\omega^1 + v\omega^2, \quad \bar{\omega}_3^2 - \omega_3^2 = u\omega^1 + \tfrac{1}{2}k\omega^2, \tag{10}$$

and the first equation of (9) gives

$$\tfrac{1}{2}kh_1 + uh_2 = 0, \quad vh_1 + \tfrac{1}{2}kh_2 = 0.$$

In order that this system of equations possesses a set of solutions h_1, h_2, not all zero, it is necessary and sufficient that

$$k^2 = 4uv. \tag{11}$$

Since the condition $u = v = 0$ signifies that Σ and $\bar{\Sigma}$ are projectively equivalent, we shall assume that u, v are not both equal to zero, and we then get

$$h_1 : h_2 = u^{\frac{1}{2}} : \varepsilon v^{\frac{1}{2}}, \quad \varepsilon^2 = 1. \tag{12}$$

Now π is determined up to a factor, so that we can put $\pi = u^{\frac{1}{2}}a_1 + \varepsilon v^{\frac{1}{2}}a_2 + ha_3$. On assuming that

$$\vartheta_0 = -(u^{\frac{1}{2}}\omega^1 + \varepsilon v^{\frac{1}{2}}\omega^2),$$

we get, from (10) and (9)

$$\left.\begin{aligned}
&\bar{\omega}_1^0 - \omega_1^0 = -u^{\frac{1}{2}}\vartheta_0, \quad \bar{\omega}_2^0 - \omega_2^0 = -\varepsilon v^{\frac{1}{2}}\vartheta_0, \\
&\bar{\omega}_3^1 - \omega_3^1 = -\varepsilon v^{\frac{1}{2}}\vartheta_0 - 2\varepsilon u^{\frac{1}{2}}v^{\frac{1}{2}}\omega^1, \quad \bar{\omega}_3^2 - \omega_3^2 = -u^{\frac{1}{2}}\vartheta_0 - 2\varepsilon u^{\frac{1}{2}}v^{\frac{1}{2}}\omega^2, \\
&\rho_2' = -\vartheta_0^2, \quad -(\vartheta_0)^2 h = \vartheta_0(\bar{\omega}_3^0 - \omega_3^0) + (du^{\frac{1}{2}} - u^{\frac{1}{2}}\omega_1^1 - \varepsilon v^{\frac{1}{2}}\omega_1^2 - h\omega_1^3) \\
&\qquad \times (\bar{\omega}_3^1 - \omega_3^1) + (\varepsilon dv^{\frac{1}{2}} - \varepsilon u^{\frac{1}{2}}\omega_2^1 - v^{\frac{1}{2}}\omega_2^2 - \varepsilon h\omega_2^3)(\bar{\omega}_3^2 - \omega_3^2).
\end{aligned}\right\} \tag{13}$$

We shall show that, if the surfaces S and $\bar{S}$ are deformable, then, for a proper choice of h, the two families of elements of contact Σ and $\bar{\Sigma}$ are also deformable.

For this purpose we remark first that we can replace the frame $\bar{A}\bar{A}_1\bar{A}_2\bar{A}_3$ attached to the point $\bar{A}$ of $\bar{S}$ by the frame $\bar{A}\bar{A}_1\bar{A}_2\bar{A}_3^*$ where $\bar{A}_3^* = \bar{A}_3 + \lambda\bar{A}$, λ being arbitrary. Under this change of frame the quantities u and v remain unaltered, while k is changed into $k + 2\lambda$. We can therefore suppose $k = -2\varepsilon u^{\frac{1}{2}}v^{\frac{1}{2}}$ in (3), and the system (3) remains in involution. Since the first four equations of (13) are

consequences of (3), it remains only to determine h from last equation of (13). From the equations of structure, we find

$$\left.\begin{aligned}(\bar{\omega}_1^0-\omega_1^0)' &= [\omega^2(\bar{\omega}_3^0-\omega_3^0)]+(v-\beta u-\alpha k)[\omega^1\omega^2],\\ (\bar{\omega}_2^0-\omega_2^0)' &= [\omega^1(\bar{\omega}_3^0-\omega_3^0)]+(-u+\alpha v+\beta k)[\omega^1\omega^2],\\ (\bar{\omega}_3^1-\omega_3^1)' &= [\bar{\omega}_3^0-\omega_3^0)\omega^1]+(u+\alpha v-\beta k)[\omega^1\omega^2],\\ (\bar{\omega}_3^2-\omega_3^2)' &= [\bar{\omega}_3^0-\omega_3^0)\omega^2]+(-v-\beta u+\alpha k)[\omega^1\omega^2].\end{aligned}\right\} \tag{14}$$

Now, the last four equations of (3) may be written

$$\left.\begin{aligned}&(\bar{\omega}_1^0-\omega_1^0)+(\bar{\omega}_3^2-\omega_3^2)=2u\omega^1,\ (\bar{\omega}_2^0-\omega_2^0)+(\bar{\omega}_3^1-\omega_3^1)=2v\omega^2,\\ &\bar{\omega}_1^0-\omega_1^0=u\omega^1-\tfrac{1}{2}k\omega^2,\ \bar{\omega}_2^0-\omega_2^0=-\tfrac{1}{2}k\omega^1+v\omega^2.\end{aligned}\right\} \tag{15}$$

On forming their bilinear covariants, we get

$$\left.\begin{aligned}&[(du-2\beta u\omega^2)\omega^1]=0,\ [(dv-2\alpha v\omega^1)\omega^2]=0,\\ &[\omega^1(\bar{\omega}_3^0-\omega_3^0)]=\left(\frac{uv_2}{k}-\beta k+u\right)[\omega^1\omega^2],\\ &[\omega^2(\bar{\omega}_3^0-\omega_3^0)]=\left(-\frac{vu_1}{k}+\alpha k-v\right)[\omega^1\omega^2],\end{aligned}\right\}$$

where u_1 and v_2 are defined by the relations

$$du=u_1\omega^1+u_2\omega^2,\ dv=v_1\omega^1+v_2\omega^2.$$

It follows that

$$\left.\begin{aligned}&du=u_1\omega^1+2\beta u\omega^2,\ dv=2\alpha v\omega^1+v_2\omega^2,\\ &\bar{\omega}_3^0-\omega_3^0=\left(\frac{vu_1}{k}-\alpha k+v\right)\omega^1+\left(\frac{uv_2}{k}-\beta k+u\right)\omega^2.\end{aligned}\right\} \tag{16}$$

From the bilinear covariant of the last equation, we get

$$u_1-v_2=4(\alpha u-\beta v).$$

It is therefore possible to write

$$du=W\omega^1+2u(\omega_1^1-\omega_0^0),\ dv=W\omega^2+2v(\omega_2^2-\omega_0^0), \tag{17}$$

where

$$W=u_1-4\alpha u=v_2-4\beta v.$$

Substituting the expressions (16), (17) in (13), and equating the corresponding co-efficients of $(\omega^1)^2, \omega^1\omega^2, (\omega^2)^2$, two of the resulting equations reduce to identities and the other determines h as follows:

$$4uvh=(2\varepsilon u^2v^{\frac{1}{2}}+2v^2u^{\frac{1}{2}}-\varepsilon vWv^{\frac{1}{2}}-uWu^{\frac{1}{2}}). \tag{18}$$

We have thus proved the

THEOREM. *From every non-ruled deformable surface S we an construct a deformable family of elements of contact by passing a plane through every point of S. Corresponding to every conjugate net of deformation of S exactly two such planes can be be constructed through every point of S.*

Geometrically, the plane π can be interpreted as follows:

The tangents to any family of curves of the conjugate net of deformation of a deformable surface S form a W-congruence. The plane associated with every point A of S mentioned above is the second focal plane of the congruence, the first focal plane being the tangent plane of S.

PROOF. Consider the direction on S defined by

$$u^{\frac{1}{2}}\omega^1 + \varepsilon v^{\frac{1}{2}}\omega^2 = 0.$$

The tangent along this direction joins the point A to the point $\varepsilon v^{\frac{1}{2}}A_1 - u^{\frac{1}{2}}A_2$. To show that π is a focal plane of the congruence formed by these tangents it is only necessary to prove that $d\pi$ passes through a fixed point of the tangent. Now we have $d\pi = \pi_1\omega^1 + \pi_2\omega^2$, where

$$\pi_1 = -u^{\frac{1}{2}}a + (\tfrac{1}{2}Wu^{-\frac{1}{2}} + \tfrac{3}{2}\alpha u^{\frac{1}{2}} - \varepsilon v^{\frac{1}{2}})a_1 + (\tfrac{3}{2}\varepsilon\alpha v^{\frac{1}{2}} - h)a_2 + (\ldots)a_3$$

$$\pi_2 = -\varepsilon v^{\frac{1}{2}}a + (\tfrac{3}{2}\beta u^{\frac{1}{2}} - h)a_1 + (\tfrac{1}{2}\varepsilon v^{-\frac{1}{2}}W + \tfrac{3}{2}\beta\varepsilon v^{\frac{1}{2}} - u^{\frac{1}{2}})a_2 + (\ldots)a_3.$$

$d\pi$ passes through a fixed point of the tangent if

$$\begin{vmatrix} \pi_1 A & \pi_1(\varepsilon v^{\frac{1}{2}}A_1 - u^{\frac{1}{2}}A_2) \\ \pi_2 A & \pi_2(\varepsilon v^{\frac{1}{2}}A_1 - u^{\frac{1}{2}}A_2) \end{vmatrix} = 0,$$

where the scalar product of a plane and a point is given by the formulae $A_\alpha a_\beta = \delta_{\alpha\beta} (\alpha, \beta = 0, 1, 2, 3)$, $\delta_{\alpha\beta}$ being the Kronecker symbol. The vanishing of the determinant can be readily verified. Hence the result is proved.

OFFICE OF THE REGISTRAR OF ... 15 MAY 1944 MADRAS

IDENTITIES IN MULTIPLICATIVE FUNCTIONS.*

BY

P. KESAVA MENON, *Annamalai University*

[Received 17 January 1943]

Let $\lambda_k(M)=k^{\nu}$, $E_k(M)=k^{\mu}$, where μ, ν are respectively the number of distinct prime factors and the total number of prime factors of M. We shall denote $E_1(M)$ by $E(M)$. Let us also define the function e by

$$e(M) = \Pi(1/r!),$$

where $M = \Pi p^r$, and Π denotes the product for all primes.

The generating series to the base p of λ_k, E_k and e are readily seen to be

$$\lambda_{k\,(p)}\,(x) = 1 + kx + k^2x^2 + \cdots = \frac{1}{1-kx},$$

$$E_{k\,(p)}\,(x) = 1 + kx + kx^2 + \ldots = \frac{1+(k-1)x}{1-x},$$

and $$e_{(p)}(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots = e^x.$$

We proceed to prove a few theorems involving the e-function.

THEOREM 1.

$$\sum_{d\,|\,M} e(d)\; e\left(\frac{M}{d}\right) \lambda_r(d)\; \lambda_s\left(\frac{M}{d}\right) = e(M)\; \lambda_{r+s}(M).$$

PROOF: We have

$$e^{rx} = 1 + \frac{rx}{1!} + \frac{r^2x^2}{2!} + \ldots = (e \times \lambda_r)_{(p)}(x).$$

* This formed part of the thesis for the M.Sc. degree of the Madras University. Also vide *Jour. Ind. Math. Soc.* 6 (1942), 137-52.

Therefore

$$\{ (e\times\lambda_r) . (e\times\lambda_s) \}_{(p)} (x) = e^{(r+s)x} = (e\times\lambda_{r+s})_{(p)}(x),$$

from which the theorem follows.

THEOREM 2.

$$\sum_{d_1 d_2 \ldots d_n = M} e(d_1)\, e(d_2) \ldots e(d_n) = e(M)\lambda_n(M).$$

For, the left side is $(e . e . \ldots . e)(M)$ or $e^n(M)$ and its generating series to the base p is e^{nx} which is also the generating series of $(e\times\lambda_n)(M)$.

If $\sigma_k(M)$ represents the number of ways of expressing M as the product of k factors (including unity), the order of the factors being taken into consideration, then

$$\sigma_k(M) = \sum_{d_1 d_2 \ldots d_k = M} E(d_1)\, E(d_2) \ldots E(d_k) = E^k(M),$$

since $E(M) = E_1(M)$ is the function which takes the value unity for all values of M.

THEOREM 3.

$$\sum_{d\mid M} e^{-1}(d)\, \sigma_{k-1}^{-1}(d) e\left(\frac{M}{d}\right) = \sigma_k(M) e(M).$$

PROOF. $(e\times\sigma_k)_{(p)}(x) = (e\times E^k)_{(p)}(x)$

$$= 1 + \frac{kx}{1!\,1!} + \frac{k(k+1)}{2!\,2!} x^2 + \frac{k(k+1)(k+2)}{3!\,3!} x^3 + \ldots,$$

since $E^k_{(p)}(x) = (1-x)^{-k} = 1 + \dfrac{kx}{1!} + \dfrac{k(k+1)}{2!} x^2 + \ldots$.

But $$1 + \frac{kx}{1!\,1!} + \frac{k(k+1)}{2!\,2!} x^2 + \ldots = \frac{1}{(k-1)!} \frac{d^{k-1}}{dx^{k-1}} (x^{k-1} e^x)$$

$$= e^x \left[1 + \binom{k-1}{1} \frac{x}{1!} + \binom{k-1}{2} \frac{x^2}{2} + \ldots + \frac{x^{k-1}}{(k-1)!} \right]$$

$$= \{ e . (e^{-1} \times E^{-(k-1)} \}_{(p)} (x),$$

since $E^{-(k-1)}_{(p)}(x) = (1-x)^{k-1}$ and $e^{-1}_{(p)}(x) = e^{-x}$.

Hence $$e\times\sigma_k = e . (e^{-1} \times \sigma_{k-1}^{-1}),$$

which is the required result.

2. Let us write

$$a(n, r) = \Sigma(m_1+1)(m_1+m_2+1)\ldots(m_1+m_2+\ldots+m_r+1),$$

where the summation is for all positive integral or zero values of $m_1, m_2, \ldots, m_r$ such that $m_1+m_2+\ldots+m_r+r \leqslant n$, and $K_n(M) = \Pi a(n, n-r)$, where r is the power of p in the relation $M = \Pi p^r$. Then we have

THEOREM 4. $\sum_{d|M} K_n(d)\, e\left(\frac{M}{d}\right) = e(M)\left\{\sigma(M)\right\}^n,$

where $\sigma(M)[=\sigma_2(M)]$ denotes the number of divisors of M, and n is a positive integer.

PROOF. The generating series to the base p of $e(M)\left\{\sigma(M)\right\}^n$ is

$$\sum_{r=0}^{\infty} \frac{(r+1)^n}{r!} x^r = \frac{1}{x}\left(x\frac{d}{dx}\right)^n [xe^x]$$

$$= e^x[a(n, 0)x^n + a(n, 1)x^{n-1} + \ldots + a(n, n)].$$

This may be proved by induction. The required result follows from this.

THEOREM 5. $\sum_{d|M} \sigma_k(d) E_k\left(\frac{M}{d}\right) = \sigma(M)\,\sigma_k(M).$

PROOF. The left-side is $(E^k . E_k)(M)$ and the right side is $(E^2 \times E^k)(M)$. Therefore we have to show that $E^k . E_k = E^2 \times E^k$. Now

$$E^k_{(p)}(x) = \frac{1}{(1-x)^k},\ E_{k(p)}(x) = \frac{1+(k-1)x}{1-x},$$

and

$$(E^2 \times E^k)_{(p)}(x) = 1 + \frac{2k}{1!}x + \frac{3k\,(k+1)}{2!}x^2 + \ldots$$

$$= \frac{1+(k-1)x}{(1-x)^{k+1}},$$

so that the result follows.

3. Next we shall consider some applications of hypergeometric series to multiplicative functions. We first of all note that if $(\alpha, m) = \alpha(\alpha+1)\ldots(\alpha+m-1)$,

$$F(\alpha, \beta, \gamma, x) = 1 + \sum_{m=1}^{\infty} \frac{(\alpha, m)(\beta, m)}{(\gamma, m)(1, m)} x^m,$$

is the generating series to any prime base p of

$$\left(E^{\alpha} \times E^{\beta} \times \frac{1}{E^{\gamma}}\right)(M).$$

THEOREM 6.*

$$\sum_{d \mid M} E^{\alpha+\beta-\gamma}(M/d) E^{\gamma-\alpha}(d) E^{\gamma-\beta}(d) / E^{\gamma}(d)$$
$$= E^{\alpha}(M) E^{\beta}(M) / E^{\gamma}(M).$$

This may be expressed as a relation between the functions $\sigma_{\alpha}(M)$, where $\sigma_{\alpha}(M)$ denotes the number of ways of expressing M as the product of α factors (including unity), the order of the factors being taken into consideration. Thus, if $0 < \alpha, \beta < \gamma < \alpha + \beta$, the theorem may be stated as

$$\sum_{d \mid M} \sigma_{\alpha+\beta-\gamma}(M/d)\, \sigma_{\gamma-\alpha}(d)\, \sigma_{\gamma-\beta}(d)/\sigma_{\gamma}(d)$$
$$= \sigma_{\alpha}(M) \sigma_{\beta}(M) / \sigma_{\gamma}(M).$$

To prove Theorem 6, we note that the left side is

$$\left\{E^{\alpha+\beta-\gamma} . \left(E^{\gamma-\alpha} \times E^{\gamma-\beta} \times \frac{1}{E^{\gamma}}\right)\right\}(M)$$

whose generating series to the base p is

$$\frac{1}{(1-x)^{\alpha+\beta-\gamma}} \times F(\gamma-\alpha, \gamma-\beta, \gamma, x) = F(\alpha, \beta, \gamma, x),$$

which is the generating series to the base p of $(E^{\alpha} \times E^{\beta} \times 1/E^{\gamma})(M)$. The required result follows.

THEOREM 7. $\sum_{d \mid M} E^{\alpha}(d) E^{\beta}(d) E^{-\alpha}(M/d) / E^{\gamma}(d)$

$= \sum_{d \mid M} T(M, d) E^{\alpha}(d) E^{\gamma-\beta}(d) / E^{\gamma}(d)$, *where*

* The particular case where $\gamma = 1$ has been proved by Dr. R. Vaidyanathaswamy. *Trans. Amer. Math. Soc.* 33 (1932).

$$T(M, N) = \Pi(-1)^n \binom{m-1}{n-1}, \; M = \Pi p^m, \; N = \Pi p^n.$$

This follows from the relation

$$(1-x)^{\alpha} F(\alpha, \beta, \gamma, x) = F\left(\alpha, \gamma-\beta, \gamma, \frac{x}{x-1}\right).$$

THEOREM 8.

$$\sum_{d \mid M} \frac{E^{\alpha}(d) E^{\beta}(d) E^{1-\alpha}(M/d) E^{1-\beta}(M/d)}{E^{\gamma}(d) E^{2-\gamma}(M/d)}$$
$$= \sum_{d \mid M} \frac{E^{\alpha+1-\gamma}(d) E^{\beta+1-\gamma}(d) E^{\gamma-\alpha}(M/d) E^{\gamma-\beta}(M/d)}{E^{2-\gamma}(d) E^{\gamma}(M/d)}.$$

Here, only those values of M are taken into consideration which do not make the denominators vanish. The theorem is a consequence of the relation

$$F(\alpha, \beta, \gamma, x) \times F(1-\alpha, 1-\beta, 2-\gamma, x)$$
$$= F(\alpha+1-\gamma, \beta+1-\gamma, 2-\gamma, x) \times F(\gamma-\alpha, \gamma-\beta, \gamma, x).$$

From limiting cases of hypergeometric series, we get identities involving the e-function. For instance we have

THEOREM 9.

$$\sum_{d \mid M} e\left(\frac{M}{d}\right) E^{\beta-\alpha}(d)\, e^{-1}(d) / E^{\beta}(d) = \frac{E^{\alpha}(M)\, e(M)}{E^{\beta}(M)}.$$

The generating series of the right side is

$$1 + \sum_{m=1}^{\infty} \frac{(\alpha, m)\, x^m}{(1, m)(\beta, m)} = G(\alpha, \beta; x), \text{ say}.$$

The theorem follows from the known result

$$G(\alpha, \beta; x) = e^x \times G(\beta-\alpha, \beta; -x).$$

ERRATA

VOLUME VI (1942)

page 176 line 2 *for* $\sum_{s=k-1}^{s=k-1+r}$ *read* $\sum_{s=k-1}^{s=k-1-r}$

" 197 " 23 *for* $>$ *read* $\geqslant$

" 198 " 7-8 *for* $\{(3/2)^s + 4.2^{-s}\}$ *read* $\{(3/2)^s + 4.2^{-s} - 1\}$

" " " 19 *for* $>$ *read* $\geqslant$

" " " 20 *for* when *read* if (implying that the most unfavourable case cannot actually arise).

NOTICE

The Editor of the *Mathematical Reviews*, Brown University, Providence, R. I., U. S. A., shall appreciate it very much if mathematicians in India can send them reprints of their papers regularly for purposes of review. Their cooperation is specially necessary under the war conditions.

Publications of the Indian Mathematical Society

1. Memoir on Cubic Transformations Associated with a Desmic System, by Dr. R. Vaidyanathaswami, pp. 92, Price Rs. 3.
2. Tables of Partitions, by Dr. Hansraj Gupta, pp. 81, Price Rs. 5.

Back Numbers of the *Journal* and of the *Student* are available, except for a few numbers out of stock.

THE MATHEMATICAL ASSOCIATION OF AMERICA

The Mathematical Association of America is devoted primarily to the interests of collegiate mathematics. It now has over two thousand individual and institutional members. There are twenty-two sections at present, representing thirty-one different States. The Association holds two national meetings per year and the sections hold one or two meetings per year. All meetings, both national and sectional are reported in the official Journal.

Application blanks for membership may be obtained from the Secretary, Prof. W.D. Cairns, Oberlin, Ohio.

THE AMERICAN MATHEMATICAL MONTHLY

Official Organ of

THE MATHEMATICAL ASSOCIATION OF AMERICA

Is the Only Journal of Collegiate Grade in the Mathematical Field in this Country.

Most of its mathematical contributions can be read and understood by those who have not specialized in mathematics beyond the Calculus.

The historical papers, which are numerous and of high grade, are based upon original research.

The QUESTIONS and DISCUSSIONS, which are timely and interesting, cover a wide variety of topics.

Surveys of the contents of recent books constitute a valuable guide to current mathematical literature.

The topics in the department of Undergraduate Mathematics Clubs have excited wide interest among those engaged in teaching mathematics.

The NEWS and NOTICES cover a wide range of interest and information

The Problems and Solutions hold the attention and activity of a large number of persons who are lovers of Mathematics for its own sake.

EDITED BY DR. R. VAIDYANATHASWAMI, READER, MADRAS UNIVERSITY AND PUBLISHED BY S. MAHADEVAN, HON. SECRETARY, PRESIDENCY COLLEGE, MADRAS.
PRINTED BY R. NARAYANASWAMI IYER
AT THE MADRAS LAW JOURNAL PRESS, MYLAPORE, MADRAS.

12.9992